CHAPTER 4

Differentiable Functions

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Differentiable Functions

1.1 The Derivative

1.1.1 *Definition and basic properties*

Definition 1.1.1. *Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ be a function, and let $c \in I$. If the limit*

$$l = \lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c},$$

exists, then we say f is differentiable at c, we call "l" the derivative of f at c, and we write $f'(c) = l$.

If f is differentiable at all $c \in I$, then we simply say that f is differentiable, and then we obtain a function $f' : I \longrightarrow \mathbb{R}$. The derivative is sometimes written as $\frac{df}{dx}$ or $\frac{d}{dx}(f(x))$. The expression $\frac{f(x) - f(c)}{x - c}$ is called the difference quotient.

The graphical interpretation of the derivative is depicted in Figure 1.2. The left-hand plot gives the line through $(c, f(c))$ and $(x, f(x))$ with slope $\frac{f(x) - f(c)}{x - c}$, that is, the so-called secant line. When we take the limit as x goes to c, we get the right-hand plot, where we see that the derivative of the function at the point c is the slope of the line tangent to the graph of f at the point $(c, f(c))$.

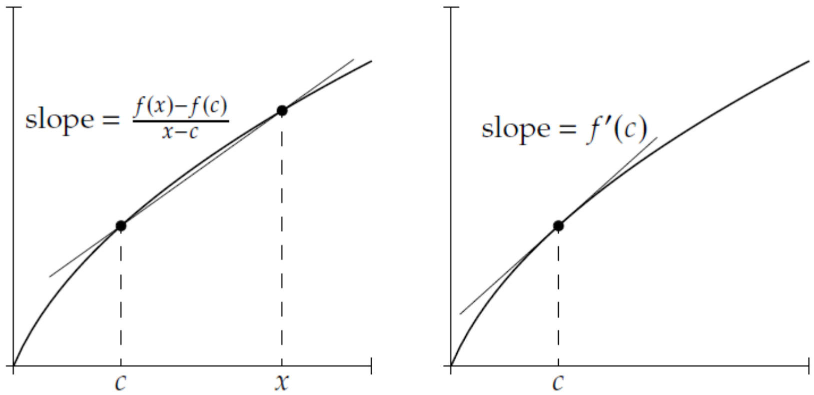

Figure 1.1: Graphical interpretation of the derivative

Example 1.1.1. *Let $f(x) = x^2$ defined on the whole real line, and let $c \in \mathbb{R}$ be arbitrary. We find that if $x \neq c$,*

$$\frac{x^2 - c^2}{x - c} = \frac{(x + c)(x - c)}{x - c} = x + c.$$

Therefore,

$$f'(c) = \lim_{x \longrightarrow c} \frac{x^2 - c^2}{x - c} = \lim_{x \longrightarrow c} (x + c) = 2c.$$

Example 1.1.2. *The function $f(x) = \sqrt{x}$ is differentiable for $x > 0$. To see this fact, fix $c > 0$, and suppose $x \neq c$ and $x > 0$. Compute*

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \longrightarrow c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \longrightarrow c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

Remark 1.1.1. *By setting $x - c = h$, the previous limit can be written in the form*

$$f'(c) = \lim_{h \longrightarrow 0} \frac{f(c + h) - f(c)}{h}.$$

Proposition 1.1.1. *Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.*

Proof 1. *We know the limits*

$$\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c}=f'(c),\qquad and \qquad \lim_{x\longrightarrow c}(x-c)=0.$$

exist. Furthermore,

$$f(x)-f(c)=\left(\frac{f(x)-f(c)}{x-c}\right)(x-c),$$

Therefore, the limit of $f(x)-f(c)$ exists and

$$\lim_{x\longrightarrow c}(f(x)-f(c))=\left(\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\longrightarrow c}(x-c)\right)=f'(c).0=0.$$

Hence $\lim_{x\longrightarrow c}f(x)=f(c)$, and f is continuous at c.

Proposition 1.1.2. *If f is differentiable over I, then f is continuous over I.*

Proposition 1.1.3. *Let I be an interval, let $f: I\longrightarrow\mathbb{R}$ and $g: I\longrightarrow\mathbb{R}$ be differentiable at $c\in I$, and let $\alpha\in\mathbb{R}$, then:*

1. ***The linearity:***

   - *Define $h: I\longrightarrow\mathbb{R}$ by $h(x)=\alpha.f(x)$. Then h is differentiable at c and $h'(c)=\alpha.f'(c)$.*
   - *Define $h: I\longrightarrow\mathbb{R}$ by $h(x)=f(x)+g(x)$. Then h is differentiable at c and $h'(c)=f'(c)+g'(c)$.*

2. ***Product rule:***
   If $h: I\longrightarrow\mathbb{R}$ is defined by $h(x)=g(x)f(x)$, then h is differentiable at c and

   $$h'(c)=f(c)g'(c)+f'(c)g(c).$$

3. ***Quotient rule:***
   If $g(x)\neq 0$ for all $x\in I$, and if $h: I\longrightarrow\mathbb{R}$ is defined by $h(x)=\dfrac{f(x)}{g(x)}$, then h is differentiable at c and

   $$h'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}.$$

1.1.2 *Chain rule*

Proposition 1.1.4. *Let I, J be intervals, let $g : I \longrightarrow J$ be differentiable at $c \in I$, and $f : J \longrightarrow \mathbb{R}$ be differentiable at $g(c)$. If $h : I \longrightarrow \mathbb{R}$ is defined by*

$$h(x) = (f \circ g)(x) = f(g(x)),$$

then h is differentiable at c and

$$h'(c) = f'(g(c))g'(c).$$

1.1.3 *Inverse function*

Proposition 1.1.5. *Let $I \subset \mathbb{R}$ be an interval, and let f be an injective and continuous function on I. If f is differentiable at point c with $f'(c) \neq 0$, then the inverse function: $f^{-1} : f(I) \longrightarrow \mathbb{R}$ is differentiable at $f(c)$ and*

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

1.2 Left and Right Derivatives

Definition 1.2.1. *Suppose $f : [a, b] \longrightarrow \mathbb{R}$. Then f is right-differentiable at $a \leq c < b$ with right derivative $f'(c^+)$ if*

$$\lim_{x \overset{>}{\longrightarrow} c} \tfrac{f(x)-f(c)}{x-c} = f'(c^+),$$

exists, and f is left-differentiable at $a < c \leq b$ with left derivative $f'(c^-)$ if

$$\lim_{x \overset{<}{\longrightarrow} c} \tfrac{f(x)-f(c)}{x-c} = f'(c^-) \text{ exists.}$$

A function is differentiable at $a < c < b$ if and only if the left and right derivatives exist at c and are equal.

Example 1.2.1. *The absolute value function* $f(x) = |x|$ *is left and right differentiable at* 0 *with left and right derivatives*

$$f'(0^+) = 1 \quad \textit{and} \quad f'(0^-) = -1.$$

These are not equal, and f *is not differentiable at* 0*.*

1.3 Successive Derivatives and Leibnitz's Rule

1.3.1 *Successive derivatives*

Let $f(x)$ be a differentiable function on an interval I. Then the derivative $f'(x)$ is a function of x and if $f'(x)$ is differentiable at x, then the derivative of $f'(x)$ at x is called second derivative of $f(x)$ at x and it is denoted by $f''(x)$ or $f^{(2)}(x)$. Proceeding in this way the $n-th$ order derivative of $f(x)$ is the derivative of the function $f^{(n-1)}(x)$ and it the denoted by $f^{(n)}(x)$.

Example 1.3.1. *1). Let* $f(x) = \sin(x)$*. Calculate* $f^{(n)}(x)$*. We have:*

$$\begin{aligned}
f^{(0)}(x) &= \sin(x), \\
f'(x) &= f^{(1)}(x) = \cos(x) = \sin(x + \frac{\pi}{2}), \\
f^{(2)}(x) &= -\sin(x) = \sin(x + \pi), \\
f^{(3)}(x) &= -\cos(x) = \sin(x + \frac{3\pi}{2}), \\
f^{(4)}(x) &= \sin(x) = \sin(x + 2\pi), \\
&\vdots \\
f^{(n)}(x) &= \sin(x + \frac{n\pi}{2}).
\end{aligned}$$

2). $f(x) = \ln x$*. Calculate* $f^{(n)}(x)$*. We have:*

$$f^{(0)}(x) = \ln x, \qquad f'(x) = \frac{1}{x},$$

$$f^{(2)}(x) = \frac{-1}{x^2}, \qquad f^{(3)}(x) = \frac{2}{x^3},$$

$$f^{(4)}(x) = \frac{-2\times 3}{x^4}, \qquad f^{(5)}(x) = \frac{2\times 3\times 4}{x^5} = \frac{4!}{x^5},$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\,\frac{(n-1)!}{x^n},\ n \in \mathbb{N}^*.$$

Definition 1.3.1. *Let I be an interval of $\mathbb{R}$ and f be a function defined on I. It is said that f is of class C^1 on I if f is differentiable on I and f' continues on I. We say that f is of class $C^n(I)$ if f is $n-$times differentiable on I and if $f^{(n)}$ continues on I.*

1.3.2 *Leibnitz formula*

Theorem 1.3.1. *If f and g be two functions each differentiable n times on I, then $f\times g$ is $n-$times differentiable on I, and:*

$$(f\times g)^{(n)} = \sum_{k=0}^{n} C_n^k\, f^{(n-k)}\, g^{(k)},\ C_n^k = \frac{n!}{k!(n-k)!}.$$

Example 1.3.2. *For $n = 2$, we have:*

$$\begin{aligned}(f\times g)^{(2)} &= C_2^0 f''g + C_2^1 f'g' + C_2^2 fg'' \\ &= f''g + 2f'g' + fg''.\end{aligned}$$

For $n = 6$, we have:

$$\begin{aligned}(f\times g)^{(6)} &= C_6^0 f^{(6)}g + C_6^1 f^{(5)}g' + C_6^2 f^{(4)}g'' + C_6^3 f^{(3)}g^{(3)} + C_6^4 f''g^{(4)} + C_6^5 f'g^{(5)} + C_6^6 fg^{(6)} \\ &= f^{(6)}g + 6f^{(5)}g' + 15f^{(4)}g'' + 20f^{(3)}g^{(3)} + 15f''g^{(4)} + 6f'g^{(5)} + fg^{(6)}.\end{aligned}$$

If $h(x) = \left(x^3 + 5x + 1\right)e^x = f(x)g(x)$, then:

$$\begin{aligned} f'(x) &= 3x^2+5, & g'(x) &= e^x, \\ f''(x) &= 6x, & g''(x) &= e^x, \\ f^{(3)}(x) &= 6, & g^{(3)}(x) &= e^x, \\ f^{(4)}(x) &= 0, & g^{(4)}(x) &= e^x, \\ f^{(n)}(x) &= 0, \ \forall n \geq 4, & g^{(n)}(x) &= e^x. \end{aligned}$$

So:

$$\begin{aligned} h^{(n)}(x) &= C_n^0 f g^{(n)} + C_n^1 f' g^{(n-1)} + C_n^2 f'' g^{(n-2)} + C_n^3 f^{(3)} g^{(n-3)} + C_n^4 f^{(4)} g^{(n-4)} + \cdots \\ &= (x^3+5x+1)e^x + n(3x^2+5)e^x + \frac{n(n-1)}{2}(6x)e^x + \frac{n(n-1)(n-2)}{6} 6e^x. \end{aligned}$$

1.4 The Mean Value Theorem

1.4.1 *Extreme values*

Definition 1.4.1. *A critical point of a function $f(x)$, is a value c in the domain of f where f is not differentiable or its derivative is* 0 *(i.e. $f'(c) = 0$).*

Definition 1.4.2. *A function f is said to have a local maximum (local minimum) at c if f is defined on an open interval I containing c and $f(x) \leq f(c)$ ($f(x) \geq f(c)$) for all $x \in I$. In either case, f is said to have a local extremum at c.*

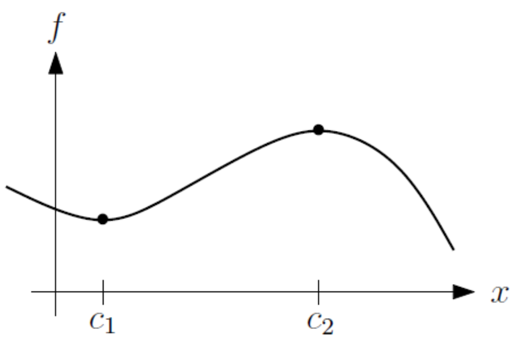

Figure 1.2: Local extrema of f

1.4.2 *Local extremum theorem*

Theorem 1.4.1. *If f has a local extremum at c and if f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose that f has a local maximum at c. Let I be an open interval containing c such that $f(x) \leq f(c)$ for all $x \in I$. Then:

$$\frac{f(c) - f(c)}{x - c} = \begin{cases} \geq 0, & if\ x \in I\ and\ x < c, \\ \leq 0, & if\ x \in I\ and\ x > c. \end{cases}$$

It follows that the left-hand derivative of f at c is ≥ 0 and the right-hand derivative is ≤ 0, hence $f'(c) = 0$. The proof for the local minimum case is similar. □

1.4.3 *Rolle's theorem*

Theorem 1.4.2. *Let f be continuous on $[a, b]$ and differentiable on $]a, b[$. If $f(a) = f(b)$, then there exists a point $c \in]a, b[$ such that $f'(c) = 0$.*

Proof. By the extreme value theorem there exist $x_m, x_M \in [a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$. If $f(x_m) = f(x_M)$, then f is a constant function and the assertion of the theorem holds trivially. If $f(x_m) \neq f(x_M)$, then either $x_m \in]a, b[$ or $x_M \in]a, b[$, and the conclusion follows from the local extremum theorem. □

1.4.4 *Mean value theorem*

Theorem 1.4.3. *If f is continuous on $[a, b]$ and differentiable on $]a, b[$, then there exists $c \in]a, b[$ such that:*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. The function $g : [a, b] \longrightarrow \mathbb{R}$ defined by:

$$g(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a),$$

is continuous on $[a, b]$ and differentiable on $]a, b[$ with

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Moreover, $g(a) = g(b) = 0$. Rolle's theorem implies that there exists $a < c < b$ such that $g'(c) = 0$, which proves the result. □

1.4.5 *Mean value inequality*

Let f be a continuous function on $[a, b]$, and differentiable on $]a, b[$. If there exists a constant M such that: $\forall\ x \in]a, b[$: $|f'(x)| \leq M$, then

$$\forall\ x,\ y \in [a, b] :\ |f(x) - f(y)| \leq M\,|x - y|.$$

According to the Mean value theorem on $[x, y]$, $\exists\ c \in]x, y[$: $f'(c) = \frac{f(x) - f(y)}{x - y}$. Then

$$|f'(c)| \leq M \Longrightarrow \left|\frac{f(x) - f(y)}{x - y}\right| \leq M \Longrightarrow M\,|x - y|\,.$$

1.5 Increasing and Derivative Functions

Let f be a continuous function on $[a, b]$, and differentiable on $]a, b[$ then:

1. $\forall\ x \in]a, b[$: $f'(x) > 0 \Longleftrightarrow f$ is strictly increasing on $[a, b]$.

2. $\forall\ x \in]a, b[$: $f'(x) < 0 \Longleftrightarrow f$ is strictly decreasing on $[a, b]$.

3. $\forall\ x \in]a, b[$: $f'(x) = 0 \Longleftrightarrow f$ is a constant.

1.6 L'Hôpital's Rule

L'Hôpital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim\limits_{x\longrightarrow c} f(x) = \lim\limits_{x\longrightarrow c} g(x) = 0$ or $\pm\infty$, and $g'(c) \neq 0$ and $\lim\limits_{x\longrightarrow c} \frac{f'(x)}{g'(x)}$ exists, then:

$$\lim_{x \longrightarrow c} \frac{f'(x)}{g'(x)} = \lim_{x \longrightarrow c} \frac{f(x)}{g(x)}.$$

Example 1.6.1. *Using L'Hopital's rule:*

1. $\lim_{x \longrightarrow 0} \frac{3x - \sin x}{x} = \lim_{x \longrightarrow 0} \frac{3 - \cos x}{1} = 2.$

2. $\lim_{x \longrightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \longrightarrow 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}.$

1.7 Convex Functions

Definition 1.7.1. *A function f is said to be convex on an interval I if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \ \forall \ t \in [0.1], \ x, \ y \in I.$$

f is concave if $-f$ is convex.

Theorem 1.7.1. *If $f :]a, b[\longrightarrow \mathbb{R}$ has an increasing derivative, then f is convex. In particular, f is convex if $f'' \geq 0$.*