CHAPTER 3

Real-Valued Functions of a Real Variable

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Real-Valued Functions of a Real Variable

1.1 Basics

1.1.1 *Definition*

Definition 1.1.1. *Let $D \subseteq \mathbb{R}$. A function f from D into $\mathbb{R}$ is a rule which associates with each $x \in D$ one and only one $y \in \mathbb{R}$. We denote*

$$\begin{aligned} f : D &\longrightarrow \mathbb{R}, \\ x &\longmapsto f(x). \end{aligned}$$

D is called the domain of the function. If $x \in D$, then the element $y \in \mathbb{R}$ which is associated with x is called the value of f at x or ***the image*** *of x under f, y is denoted by $f(x)$.*

1.1.2 *Graph of a function*

Definition 1.1.2. *Each couple $(x, f(x))$ corresponds to a point in the $xy-$plane. The set of all these points forms a curve called the graph of function f.*

$$G_f = \{(x, y) |\ x \in D,\ y = f(x)\}.$$

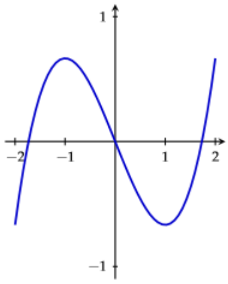


Figure 1.1: Graph of function $f(x) = 1/3x^3 - x$ in interval $[-2, 2]$.

1.1.3 *Operations on functions*

Arithmetic

Let $f, g : D \longrightarrow \mathbb{R}$ be tow functions, then:

1. $(f \pm g)(x) = f(x) \pm g(x), \ \forall x \in D,$
2. $(f.g)(x) = f(x).g(x), \ \forall x \in D,$
3. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \ g(x) \neq 0, \ \forall x \in D,$
4. $(\lambda.f)(x) = \lambda.f(x), \ \forall x \in D, \ \lambda \in \mathbb{R}.$

Composition

Let $f : D \longrightarrow \mathbb{R}$ and let $g : E \longrightarrow \mathbb{R}$, if $f(D) \subseteq E$, then g composed with f is the function $g \circ f : D \longrightarrow \mathbb{R}$ defined by $g \circ f = g[f(x)]$.

Restriction

We say that g is a restriction of the function f if:

$$g(x) = f(x) \text{ and } D(g) \subseteq D(f).$$

Example 1.1.1. $f(x) = \ln|x|$, *and* $g(x) = \ln x$, $\forall x \in]0, +\infty[$: $g(x) = f(x)$, *and* $D(g) \subseteq D(f)$.

1.1.4 *Bounded functions*

Definition 1.1.3. *Let $f : D \longrightarrow \mathbb{R}$ be a function, then:*

- *We say that f is **bounded below** on its domain $D(f)$ if*

$$\forall x \in D(f),\ \exists\, m \in \mathbb{R} :\ m \leq f(x).$$

- *We say that f is **bounded above** on its domain $D(f)$ if*

$$\forall x \in D(f),\ \exists\, M \in \mathbb{R} :\ f(x) \geq M.$$

- *Function is **bounded** if it is bounded below and above.*

Definition 1.1.4. *Let $f, g : D \longrightarrow \mathbb{R}$ be two functions, then:*

- $f \geq g$ *si* $\forall x \in D :\ f(x) \geq g(x)$.
- $f \geq 0$ *si* $\forall x \in D :\ f(x) \geq 0$.
- $f > 0$ *si* $\forall x \in D :\ f(x) > 0$.
- *f is said to be constant over D if* $\exists a \in \mathbb{R},\ \forall x \in D :\ f(x) = a$.
- *f is said to be zero over D if* $\forall x \in D :\ f(x) = 0$.

1.1.5 *Monotone functions*

Definition 1.1.5. *Consider $f : D(f) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. For all $x, y \in D$, we have:*

- *f is **increasing** (or **strictly increasing**) over D if: $x \leq y \Rightarrow f(x) \leq f(y)$, (or $x < y \Rightarrow f(x) < f(y)$).*
- *f is **decreasing** (or **strictly decreasing**) over D if: $x \leq y \Rightarrow f(x) \geq f(y)$, (or $x < y \Rightarrow f(x) > f(y)$).*

- *f is **monotone** (or **strictly monotone**) over D if f is increasing or decreasing (strictly increasing or strictly decreasing).*

Proposition 1.1.1. *A sum of two increasing (decreasing) functions is an increasing (decreasing) function.*

1.1.6 *Even and odd functions*

Definition 1.1.6.
- *We say that function $f : D(f) \longrightarrow \mathbb{R}$ is **even** if*

$$\forall x \in D(f) : \ f(-x) = f(x).$$

- *We say that function $f : D(f) \longrightarrow \mathbb{R}$ is **odd** if*

$$\forall x \in D(f) : \ f(-x) = -f(x).$$

Remark 1.1.1.
1. *Graph of an even function is symmetric with, respect to the y axis.*
2. *Graph of an odd function is symmetric with, respect to the origin.*
3. *Domain of an even or odd function is always symmetric with respect to the origin.*

1.1.7 *Periodic functions*

Definition 1.1.7. *A function $f : D(f) \longrightarrow \mathbb{R}$ is called **periodic** if $\exists\, T \in \mathbb{R}^*$ such that:*

1. $x \in D(f) \Rightarrow x \pm T \in D(f)$,
2. $x \in D(f) : f(x \pm T) = f(x)$.

Number T is called a period of f.

1.2 Limits of Functions

1.2.1 *Definition*

Definition 1.2.1. *A set $U \subset \mathbb{R}$ is a neighborhood of a point $x \in \mathbb{R}$ if:*

$$]x-\delta, x+\delta[\subset U,$$

for some $\delta > 0$. The open interval $]x-\delta, x+\delta[$ is called a $\delta-$neighborhood of x.

Example 1.2.1. *If $a < x < b$ then the closed interval $[a, b]$ is a neighborhood of x, since it contains the interval $]x-\delta, x+\delta[$ for sufficiently small $\delta > 0$. On the other hand, $[a, b]$ is not a neighborhood of the endpoints a, b since no open interval about a or b is contained in $[a, b]$.*

Definition 1.2.2. *Let f be a function defined in the neighborhood of x_0 except perhaps at x_0. A number $l \in \mathbb{R}$ is the limit of f at x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \neq x_0 :\ |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Notation: $\lim_{x \to x_0} f(x) = l$.

Example 1.2.2. *Let*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow 5x - 3 \end{aligned}$$

Show that $\lim_{x \to 1} f(x) = 2$.
By definition: $\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \neq 1 :\ |x - 1| < \delta \Rightarrow |f(x) - l| < \varepsilon$. *So we have:*

$$\forall \varepsilon > 0,\ |5x - 3 - 2| < \varepsilon \Rightarrow |5x - 5| < \varepsilon \Rightarrow 5\,|x - 1| < \varepsilon.$$

Then: $|x - 1| < \dfrac{\varepsilon}{5}$, *so* $\exists\, \delta = \dfrac{\varepsilon}{5} > 0$ *such that* $\lim_{x \to 1} f(x) = 2$.

1.2.2 *Right and left limits*

Definition 1.2.3. *Let f be a function defined in the neighborhood of x_0.*

- *We say that f has a limit l to the right of x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x_0 < x < x_0 + \delta \Rightarrow |f(x) - l| < \varepsilon.$$

We write $\lim_{x \to x_0^+} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) = l.$

- *We say that f has a limit l to the left of x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x_0 - \delta < x < x_0 \Rightarrow |f(x) - l| < \varepsilon.$$

We write $\lim_{x \to x_0^-} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = l.$

- *If f admits a limit at the point x_0 then:*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = l.$$

Example 1.2.3. *Consider the integer part function at the point $x = 2$.*

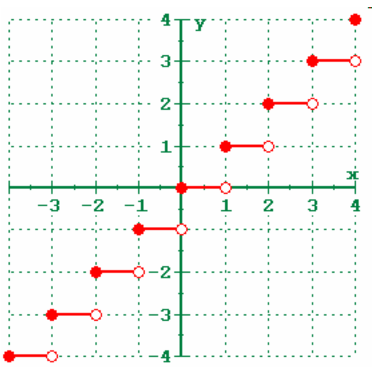


Figure 1.2: Graph of function $f(x) = E(x)$.

- *Since* $x \in]2, 3[$, *we have:* $E(x) = 2$, *and* $\lim_{x \to 2^+} E(x) = 2.$

- *Since* $x \in]1, 2[$, *we have:* $E(x) = 1$, *and* $\lim_{x \to 2^-} E(x) = 1.$

Since these two limits are different, it is deduced that $f(x) = E(x)$ has no limit at $x = 2$.

Theorem 1.2.1. *If $\lim_{x\to x_0} f(x)$ exists, then it is unique. That is, f can have only one limit at x_0.*

Proposition 1.2.1. *If $\lim_{x\to x_0} f(x) = l$, and $\lim_{x\to x_0} g(x) = l'$, $l,\ l' \in \mathbb{R}$, then:*

1. $\lim_{x\to x_0}(\lambda.f)(x) = \lambda.\lim_{x\to x_0} f(x) = \lambda.l$, $\forall \lambda \in \mathbb{R}$.
2. $\lim_{x\to x_0}(f+g)(x) = l + l'$, *and* $\lim_{x\to x_0}(f\times g)(x) = l \times l'$.
3. *If* $l \neq 0$, *then* $\lim_{x\to x_0}\left(\frac{1}{f(x)}\right) = \frac{1}{l}$.
4. $\lim_{x\to x_0} g \circ f = l'$.
5. $\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{l}{l'}$, $l' \neq 0$.
6. $\lim_{x\to x_0} |f(x)| = |l|$.
7. *If* $f \leq g$, *then* $l \leq l'$.
8. *If* $f(x) \leq g(x) \leq h(x)$, *and* $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l \in \mathbb{R}$, *then* $\lim_{x\to x_0} g(x) = l$.

1.2.3 *Infinite limits*

Definition 1.2.4. *(Limits as $\longrightarrow \pm\infty$)*

- $\lim_{x\to+\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0,\ \exists\, A > 0,\ \forall x \in \mathbb{R}:\ x > A \Rightarrow |f(x) - l| < \varepsilon$.
- $\lim_{x\to-\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0,\ \exists\, A > 0,\ \forall x \in \mathbb{R}:\ x < -A \Rightarrow |f(x) - l| < \varepsilon$.
- $\lim_{x\to+\infty} f(x) = +\infty$ *(resp:* $\lim_{x\to+\infty} f(x) = -\infty$*)* $\Leftrightarrow \forall A > 0,\ \exists\, B > 0,\ \forall x \in \mathbb{R}:\ x > B \Rightarrow f(x) > A$, *(resp:* $\forall A > 0,\ \exists\, B > 0,\ \forall x \in \mathbb{R}:\ x > B \Rightarrow f(x) < -A$*).*
- $\lim_{x\to-\infty} f(x) = +\infty$ *(resp:* $\lim_{x\to-\infty} f(x) = -\infty$*)* $\Leftrightarrow \forall A > 0,\ \exists\, B > 0,\ \forall x \in \mathbb{R}:\ x < -B \Rightarrow f(x) > A$, *(resp:* $\forall A > 0,\ \exists\, B > 0,\ \forall x \in \mathbb{R}:\ x < -B \Rightarrow f(x) < -A$*).*

1.3 Continuous Functions

1.3.1 *Continuity at a point*

Definition 1.3.1. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subset \mathbb{R}$, and let $x_0 \in I$. Then f is continuous at x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \in I :\ |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

In another word: $\lim_{x \to x_0} f(x) = f(x_0)$.

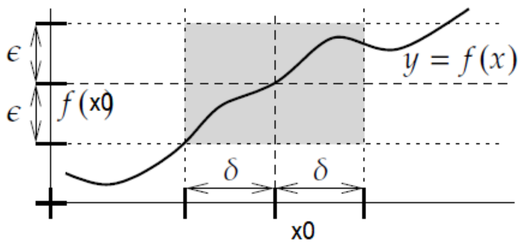


Figure 1.3: For $|x - x_0| < \delta$, the graph of $f(x)$ should be within the gray region.

A function $f : I \longrightarrow \mathbb{R}$ is continuous on a set $J \subset I$ if it is continuous at every point in J, and continuous if it is continuous at every point of its domain I.

1.3.2 *Left and right continuity*

Definition 1.3.2. *Let $f : I \longrightarrow \mathbb{R}$, we say that:*

- *f is continuous on the right of $x_0 \in I$ if:* $\lim_{x \xrightarrow{>} x_0} f(x) = f(x_0)$.
- *f is continuous on the left of $x_0 \in I$ if:* $\lim_{x \xrightarrow{<} x_0} f(x) = f(x_0)$.
- *f is continuous on $x_0 \in I$ if:* $\lim_{x \xrightarrow{>} x_0} f(x) = \lim_{x \xrightarrow{>} x_0} f(x) = f(x_0)$.

Example 1.3.1. *Let*

$$\begin{aligned} f : \mathbb{R}_+^* &\longrightarrow \mathbb{R}_+ \\ x &\longrightarrow f(x) = \sqrt{x} \end{aligned}$$

We show that f is continuous at every point $x_0 \in \mathbb{R}_+^*$, i.e.

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \in \mathbb{R}_+^* :\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon,$$

then, $\forall \varepsilon > 0$ we have:

$$\begin{aligned} |f(x) - f(x_0)| < \varepsilon &\Rightarrow \left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon \\ &\Rightarrow \left|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| < \varepsilon \\ &\Rightarrow \frac{|x - x_0|}{\sqrt{x} - \sqrt{x_0}} < \varepsilon \Rightarrow |x - x_0| < \varepsilon\left(\sqrt{x} - \sqrt{x_0}\right). \end{aligned}$$

So $\exists\, \delta = \varepsilon\left(\sqrt{x} - \sqrt{x_0}\right)$ such that: $|f(x) - f(x_0)| < \varepsilon$, then f is continous at x_0.

1.3.3 *Properties of continuous functions*

Theorem 1.3.1. *If $f, g : I \longrightarrow \mathbb{R}$ are continuous function at $x_0 \in I$ and $k \in \mathbb{R}$, then $k.f, f + g$, and $f.g$ are continuous at x_0. Moreover, if $g(x_0) \neq 0$ then f/g is continuous at x_0.*

Theorem 1.3.2. *Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ where $f(I) \subset J$. If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0) \in J$, then $g \circ f : I \longrightarrow \mathbb{R}$ is continuous at x_0.*

Proposition 1.3.1. *Let $f : I \longrightarrow \mathbb{R}$ and $x_0 \in I$, then:*

f is continuous at $x_0 \Longrightarrow$ for any sequence (u_n) that converges to x_0, the sequence $(f(u_n))$ converges to $f(x_0)$.

1.3.4 *Continuous extension to a point*

Definition 1.3.3. *Let f be a function defined in the neighborhood of x_0 except at x_0 ($x_0 \notin D_f$), and $\lim_{x\to x_0} f(x) = l$. Then the function which is defined by*

$$\tilde{f} = \begin{cases} f(x) & : \ x \neq x_0, \\ l & : \ x = x_0. \end{cases}$$

is continuous at x_0. $\tilde{f}$ is the continuous extension of f at x_0.

Example 1.3.2. *Show that:*

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \ x \neq 2.$$

has a continuous extension to $x = 2$, and find that extension.

Solution:

$\lim_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+3)}{(x-2)(x+2)} = \dfrac{5}{4}$, *exists. So f has a continuous extension at $x = 2$ defined by*

$$\tilde{f} = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - 4} & : \ x \neq 2, \\ \\ \dfrac{5}{4} & : \ x = 2. \end{cases}$$

1.3.5 *Discontinuous functions*

When f is not continuous at x_0, we say f is discontinuous at x_0, or that it has a discontinuity at x_0.

We say that the function f is not continuous in the following cases:

1. If f is not defined at x_0, then f is discontinuous at x_0.
2. If f defined in the neighborhood of x_0, then f is discontinuous at x_0 if

$$\exists\, \varepsilon > 0, \ \forall \delta > 0, \ \exists\, x \in I : \ |x - x_0| < \delta, \ and \ |f(x) - f(x_0)| \geq \varepsilon.$$

3. If $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) \neq \lim_{x \overset{<}{\longrightarrow} x_0} f(x)$, then f is discontinuous at x_0, and x_0 is a discontinuous point of the first kind.
4. If one of the two limits $\lim_{x \overset{>}{\longrightarrow} x_0} f(x)$, $\lim_{x \overset{<}{\longrightarrow} x_0} f(x)$ or both limits does not exist or are not finite, then f is discontinuous at x_0, and x_0 is a discontinuous point of the second kind.
5. If $\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) \neq f(x_0)$, then f is discontinuous at x_0.

1.3.6 *Uniform continuity*

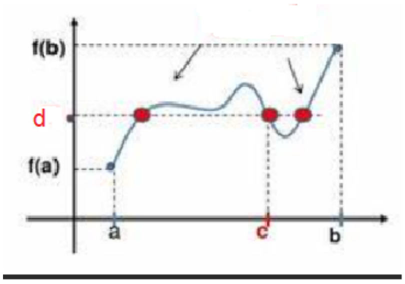
Definition 1.3.4. *Let $f : I \longrightarrow \mathbb{R}$. Then f is uniformly continuous on I if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x',\ x'' \in I :\ |x' - x''| < \delta \Longrightarrow |f(x') - f(x'')| < \varepsilon.$$

Lemma 1.3.1. *Let $f : I \longrightarrow \mathbb{R}$ be a function. If f is uniformly continuous, then f is continuous.*

1.3.7 *The intermediate value theorem*

Theorem 1.3.3. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded interval. Then for every d strictly between $f(a)$ and $f(b)$ there is a point $a < c < b$ such that $f(c) = d$.*



Corollary 1.3.1. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded interval. If $f(a).f(b) < 0$, then there is a point $a < c < b$ such that $f(c) = 0$.*

Corollary 1.3.2. *Let $f : D \longrightarrow \mathbb{R}$ is a continuous function and $I \subseteq D$ is an interval, then $f(I)$ is an interval.*

Theorem 1.3.4. *Let $I = [a, b]$ be a closed interval, and $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Theorem 1.3.5. *Any continuous function on a closed interval $[a, b]$ is bounded on $[a, b]$, i.e:* $\sup\limits_{[a,b]} |f(x)| < +\infty$.

Remark 1.3.1. *1. The image by a continuous function of a closed interval of $\mathbb{R}$ is a closed interval.*

2. *If I is not closed then the interval $f(I)$ is not necessarily of the nature of I. For example: $f(x) = x^2$, then $f(]-1,1[) = [0,1[$.*

1.3.8 *Fixed point theorem*

Definition 1.3.5. *Let $f : I \longrightarrow I$ and let $\dot{x} \in I$, we say that $\dot{x} \in I$ is a fixed point of f if: $f(\dot{x}) = \dot{x}$.*

Theorem 1.3.6. *Let $f : [a,b] \longrightarrow [a,b]$ be a continuous function, then f admits at least one fixed point in $[a,b]$ i.e: $\exists \dot{x} \in [a,b]$ such that $f(\dot{x}) = \dot{x}$.*