Solution of series $N°2$

Exercise 1:

1. $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}\cdots\cdots P(n)$

   For $n=1 \Longrightarrow 1=\dfrac{1.(1+1)}{2}$ is true.

   For $n\geq 2$: assume that $P(n)$ is true, and show that $P(n+1)$ is true, this means showing that if $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ then $1+2+3+\cdots+(n+1)=\dfrac{(n+1)(n+2)}{2}$.

   We have: $1+2+3+\cdots+n+n+1=\dfrac{n(n+1)}{2}+(n+1)=\dfrac{n(n+1)+2(n+1)}{2}=\dfrac{(n+1)(n+2)}{2}$.

   Then $P(n)$ is true, therfore $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$.

2. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

   For $n=1 \Longrightarrow 1=\dfrac{1.(2)(3)}{6}$ is true.

   For $n\geq 2$: assume that $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$, and show that $1^2+2^2+3^2+\cdots+(n+1)^2=\dfrac{(n+1)(n+2)(2n+3)}{6}$.

   We have:

$$\begin{aligned}1^2+2^2+\cdots+n^2+(n+1)^2 &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2\\ &= \frac{n(n+1)(2n+1)+6(n+1)^2}{6}\\ &= \frac{(n+1)[2n^2+n+6n+6]}{6}\\ &= \frac{(n+1)[2n^2+7n+6]}{6}\\ &= \frac{(n+1)(n+2)(2n+3)}{6}.\end{aligned}$$

   Then: $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Exercise 2:

1. $U_n = \dfrac{\cos n - 2}{n^4}$, $\forall n \in \mathbb{N}^*$.

For all $n \in \mathbb{N}^*$:

$$\begin{array}{rcll} -1 & \leq & \cos n & \leq 1 \\ -3 & \leq & \cos n - 2 & \leq -1 \\ \dfrac{-3}{n^4} & \leq & \dfrac{\cos n - 2}{n^4} & \leq \dfrac{-1}{n^4} \end{array}$$

Since $\lim\limits_{n\to\infty} \dfrac{-3}{n^4} = \lim\limits_{n\to\infty} \dfrac{-1}{n^4} = 0$, then $\lim\limits_{n\to\infty} U_n = 0$.

2. $V_n = \dfrac{3n + 5(-1)^n}{2n+1}$, $\forall n \in \mathbb{N}$.

For all $n \in \mathbb{N}$, we have

$$\frac{3n + 5(-1)^n}{2n+1} = \frac{3n}{2n+1} + \frac{5(-1)^n}{2n+1} = \frac{3}{2(1+\frac{1}{n})} + \frac{5(-1)^n}{2n+1}.$$

On the one hand since $\lim\limits_{n\to\infty} 1 + \dfrac{1}{n} = 1$, then $\lim\limits_{n\to\infty} \dfrac{3}{2(1+\frac{1}{n})} = \dfrac{3}{2}$. On the other hand since $(-1)^n$ is bounded, and $\lim\limits_{n\to\infty} \dfrac{5}{2n+1} = 0$. We deduce that $\lim\limits_{n\to\infty} \dfrac{5(-1)^n}{2n+1} = 0$. So $\lim\limits_{n\to\infty} V_n = \dfrac{3}{2}$.

3. $W_n = (-1)^n(\dfrac{n+1}{n})$, $\forall n \in \mathbb{N}^*$.

We have: $W_n = (-1)^n(\dfrac{n+1}{n}) = (-1)^n + \dfrac{(-1)^n}{n}$, since $(-1)^n$ is bounded and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, then $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n} = 0$. Also $(-1)^n$ does not admit a limits, therfore we consider the subsequences of even and odd ranks respectively $(W_{2n})_{n\in\mathbb{N}^*}$, and $(W_{2n+1})_{n\in\mathbb{N}^*}$, so for all $n \in \mathbb{N}^*$ we have:

$$\begin{array}{rcl} W_{2n} & = & (-1)^{2n} + \dfrac{(-1)^{2n}}{2n} = 1 + \dfrac{1}{2n} \underset{n\to\infty}{\longrightarrow} 1 \\ W_{2n+1} & = & (-1)^{2n+1} + \dfrac{(-1)^{2n+1}}{2n+1} = -1 - \dfrac{1}{2n+1} \underset{n\to\infty}{\longrightarrow} -1. \end{array}$$

So the sequence $(W_n)_{n\in\mathbb{N}^*}$ admits two subsequences that converge to different limits, and therefore it is not convergent.

Exercise 3:

$$\begin{cases} u_0 \in \left]0,1\right], \\ u_{n+1} = \dfrac{u_n}{2} + \dfrac{(u_n)^2}{4}. \end{cases}$$

1. We show that: $\forall n \in \mathbb{N},\ u_n \ > 0$. (reasoning by induction)

   For $n = 0$, we have $u_0 \in \left]0,1\right]$, then $u_n \ > 0$.

   For $n \geq 1$, we assume that $u_n > 0$ and we show that $u_{n+1} > 0$. We have $u_n > 0$, so: $\dfrac{u_n}{2} > 0$, and $\dfrac{(u_n)^2}{4} > 0$, therfore: $u_{n+1} = \dfrac{u_n}{2} + \dfrac{(u_n)^2}{4} > 0$. Then $\forall n \in \mathbb{N},\ u_n > 0$.

2. We show that: $\forall n \in \mathbb{N},\ u_n \ \leq 1$:

   For $n = 0$, we have $u_0 \in \left]0,1\right]$, then $u_n \ \leq 1$.

   For $n \geq 1$, we assume that $u_n \leq 1$ and we show that $u_{n+1} \leq 1$.

   We have $0 < u_n \leq 1$, then

$$u_{n+1} = \frac{u_n}{2} + \frac{(u_n)^2}{4} \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{2} \leq 1.$$

   So $\forall n \in \mathbb{N},\ u_n \leq 1$.

3. We calculate:

$$u_{n+1} - u_n = \frac{u_n}{2} + \frac{(u_n)^2}{4} - u_n = -\frac{u_n}{2} + \frac{(u_n)^2}{4} = \frac{u_n}{4}(-2 + u_n).$$

   Since $0 < u_n \leq 1$, we get $-2 + u_n < 0$, then $u_{n+1} - u_n < 0$. It shows that the sequence is strictly decreasing.

4. The sequence is strictly decreasing and bounded below by 0, so it converges to a limit noted l and verified

$$
\begin{aligned}
l = \frac{l}{2} + \frac{l^2}{4} &\iff 0 = -\frac{l}{2} + \frac{l^2}{4} \\
&\iff -2l + l^2 = 0 \\
&\iff l(-2+l) = 0
\end{aligned}
$$

so $l = 0$ or $l = 2$. Therefore $l = 0$.

Exercise 4:

$\forall n \in \mathbb{N}^*$, we have: $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$, and $v_n = u_n + \frac{1}{n}$, we show that $(u_n)_{n\in\mathbb{N}^*}$, and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent:

1. $$
\begin{aligned}
u_{n+1} - u_n &= \sum_{k=1}^{n+1} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \\
&= \frac{1}{(n+1)^2} > 0
\end{aligned}
$$
therfore $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2. $$
\begin{aligned}
v_{n+1} - v_n &= u_{n+1} + \frac{1}{n+1} - u_n - \frac{1}{n} \\
&= \frac{1}{(n+1)^2} + \frac{1}{n+1} - \frac{1}{n} \\
&= \frac{n + n(n+1) - (n+1)^2}{n(n+1)^2} \\
&= \frac{-1}{n(n+1)^2} < 0
\end{aligned}
$$
therfore $(v_n)_{n\in\mathbb{N}^*}$ is decreasing.

3. $\lim_{n\to\infty} u_n - v_n = \lim_{n\to\infty} u_n - u_n - \frac{1}{n} = \lim_{n\to\infty} \frac{-1}{n} = 0.$

   So $(u_n)_{n\in\mathbb{N}^*}$, and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.

Exercise 5:

$\forall n \in \mathbb{N}^*$ we have: $u_n = \frac{E(\sqrt{n})}{n}$, we show that $\lim_{n\to\infty} u_n = 0$.

Assume that $P = E(\sqrt{n})$, then $\forall n \in \mathbb{N}^*$ we have:

$$
P \leq \sqrt{n} < P + 1 \Longrightarrow P^2 \leq n < (P+1)^2,
$$

therfore: $\frac{1}{(P+1)^2} < \frac{1}{n} \leq \frac{1}{P^2} \cdots\cdots (*)$.

We multiply $(*)$ by $P = E(\sqrt{n}) > 0$ (because $n \geq 1$), we get:

$$\frac{P}{(P+1)^2} < \frac{P}{n} \leq \frac{P}{P^2} \Longrightarrow \frac{E(\sqrt{n})}{(E(\sqrt{n})+1)^2} < \frac{E(\sqrt{n})}{n} \leq \frac{1}{E(\sqrt{n})}.$$

When $n \longrightarrow +\infty$, $E(\sqrt{n}) \longrightarrow +\infty$, then $\lim\limits_{n\to\infty} \frac{E(\sqrt{n})}{n} = 0$.

Exercise 6:

1. $u_n = \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{(n+1)(n+2)}$.

$$\begin{aligned} \lim_{n\to+\infty} u_n &= \lim_{n\to+\infty} \left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right] \\ &= \lim_{n\to+\infty} \left(\frac{1}{2}-\frac{1}{n+2}\right) \\ &= \frac{1}{2}. \end{aligned}$$

2. $v_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}$.

$$\begin{aligned} \lim_{n\to+\infty} v_n &= \lim_{n\to+\infty} \frac{1}{n^2}(1+2+\cdots+n-1) \\ &= \lim_{n\to+\infty} \frac{1}{n^2}\frac{n(n-1)}{2} \\ &= \frac{1}{2}. \end{aligned}$$

3. $w_n = \frac{\ln(n+1)}{\ln n}$.

$$\begin{aligned} \lim_{n\to+\infty} w_n &= \lim_{n\to+\infty} \frac{\ln\left[n(1+\frac{1}{n})\right]}{\ln\ n} \\ &= \lim_{n\to+\infty} \frac{\ln\ n + \ln\ (1+\frac{1}{n})}{n} \\ &= \lim_{n\to+\infty} 1 + \frac{\ln(1+\frac{1}{n})}{\ln\ n} = 1. \end{aligned}$$

Exercise 7:

$\forall n \in \mathbb{N}^* :\ u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$.

1. We show that $\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$:

   we have: $\forall n \in \mathbb{N}^* :\ n \geq n-1 \Longrightarrow n^2 \geq n(n-1)$, so

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

2. We show that $(u_n)_{n\geq 1}$ is bounded above by 2:

we have: $\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$, then

$$\frac{1}{2^2} \leq 1 - \frac{1}{2},\ \frac{1}{3^2} \leq \frac{1}{2} - \frac{1}{3},\ \cdots,\ \frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$$

therefore:

$$\begin{aligned} 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} &\leq 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ u_n &\leq 2 - \frac{1}{n} < 2 \end{aligned}.$$

So $(u_n)_{n\geq 1}$ is bounded above by 2.

3. We show that $(u_n)_{n\geq 1}$ is increasing:

$$\begin{aligned} u_{n+1} - u_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{(n+1)^2} - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \\ &= \frac{1}{(n+1)^2} > 0. \end{aligned}$$

Then $(u_n)_{n\geq 1}$ is increasing.

4. $(u_n)_{n\geq 1}$ is increasing and bounded above by 2, so $(u_n)_{n\geq 1}$ is convergent.

Exercise 8:

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \frac{1}{6}\, u_n^2 + \frac{3}{2} \end{cases}$$

1. We show that $\forall n \in \mathbb{N}^*,\ u_n > 0$.

- For $n = 1 \Longrightarrow u_1 = \frac{1}{6}\, u_0^2 + \frac{3}{2} = \frac{3}{2} > 0.$
- For $n \geq 2 \Longrightarrow$, we assume that $u_n > 0$ and we prove that $u_{n+1} > 0$. We have $u_n > 0$, then $\frac{1}{6}u_n^2 > 0$, therefore: $\frac{1}{6}u_n^2 + \frac{3}{2} > \frac{3}{2} > 0$, so $u_{n+1} > 0 \Longrightarrow \forall n \in \mathbb{N}^*,\ u_n > 0.$

2. If the sequence u_n admits a limit l then:

$$l = \frac{1}{6}\, l^2 + \frac{3}{2} \iff l^2 - 6l + 9 = 0$$
$$\iff (l-3)^2 = 0$$
$$\iff l = 3.$$

3. We show that $\forall n \in \mathbb{N},\ u_n < 3$: (reasoning by induction)

- For $n = 0$, we have $u_0 = 0 < 3$.
- For $n \geq 1$, we assume that $u_n < 3$, and we prove that $u_{n+1} < 3$. We have

$$u_n < 3 \implies u_n^2 < 9$$
$$\implies \frac{1}{6}\, u_n^2 + \frac{3}{2} < 3.$$

So $\forall n \in \mathbb{N},\ u_n < 3$.

4. $u_{n+1} - u_n = \frac{1}{6}\,(u_n - 3)^2 > 0$, the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly increasing, and since it is bounded by 3, it therefore converges to a limit l, such that

$$l = \frac{1}{6} l^2 + \frac{3}{2} \Longrightarrow l = 3.$$