Commande adaptative

3.2. Introduction

D'une point de vue pratique, on regroupe sous les termes de commande adaptative un ensemble de concepts et de techniques utilisés pour l'ajustement automatique et en temps réel des régulateur mis en œuvre dans la commande d'un système lorsque les paramètres de ce système sont difficile à déterminer ou variant avec le temps.

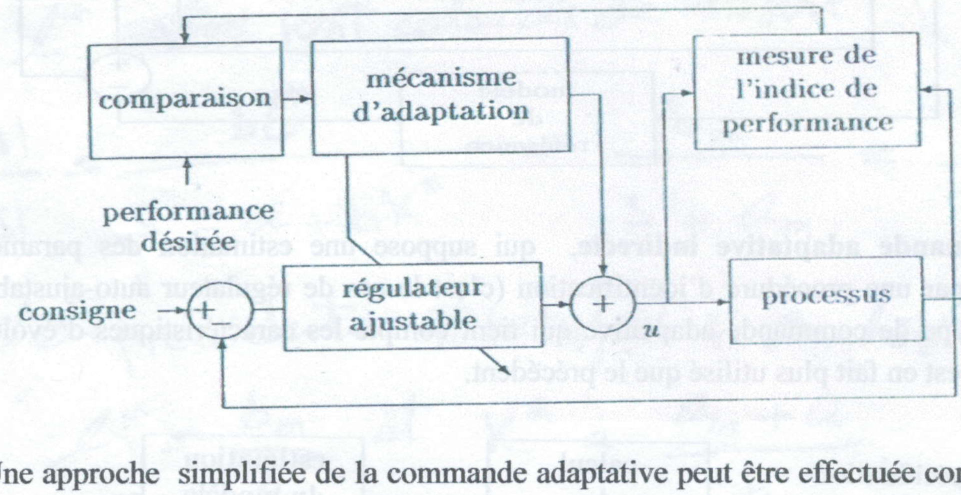
La synthèse d'une commande adaptative impose le plus souvent les phases suivantes :

- Spécification des performances désirées, (temps de réponse, localisation de pôles, minimisation d'énergie de commande, …), on cherche, lorsque c'est possible, à les caractériser par un indice de performances.
- Définition de la structure de commande ou de type de régulateur qui sera utilisé en vue de réaliser les performances souhaitées.
- Conception du mécanisme d'adaptation qui permettra d'ajuster de façons optimale les paramètres du régulateur utilisé.

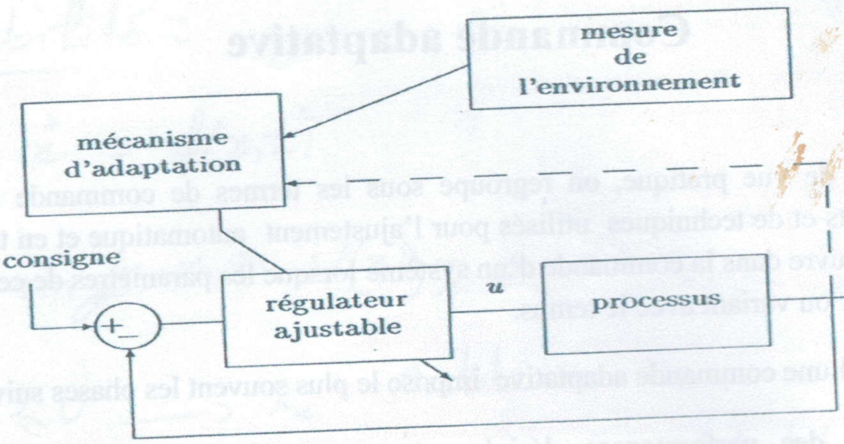
Les taches qui incombent au mécanisme d'adaptation sont les suivantes :

- Ajustement automatique des régulateurs et optimisation de leurs paramètres en les divers points de fonctionnement du système.
- Maintenance des performances exigées en cas de variation des paramètres du système.
- Détection des variations anormales des caractéristiques du système.

Le principe de mis en œuvre d'un système de commande adaptative est représenté à la figure suivante :



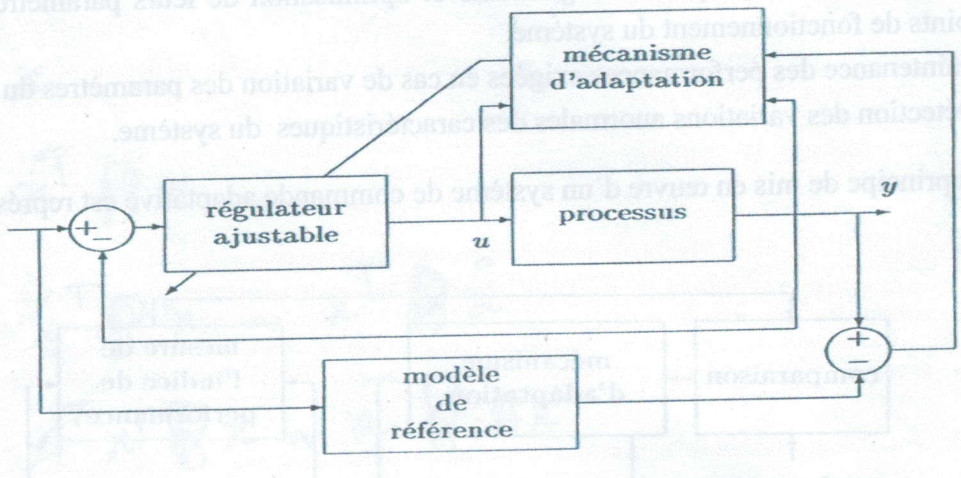
Une approche simplifiée de la commande adaptative peut être effectuée comme dans le cas des régulateurs **à gain programmé.** Dans ce cas, les valeurs des paramètres sont ajustées en fonction de l'évolution de variables caractéristique de l'environnement et du système lui-même. L'adaptation se fait alors par lecture dans une table prédéfinissant les valeurs de réglage en fonction des mesures disponible sur l'environnement et le système.



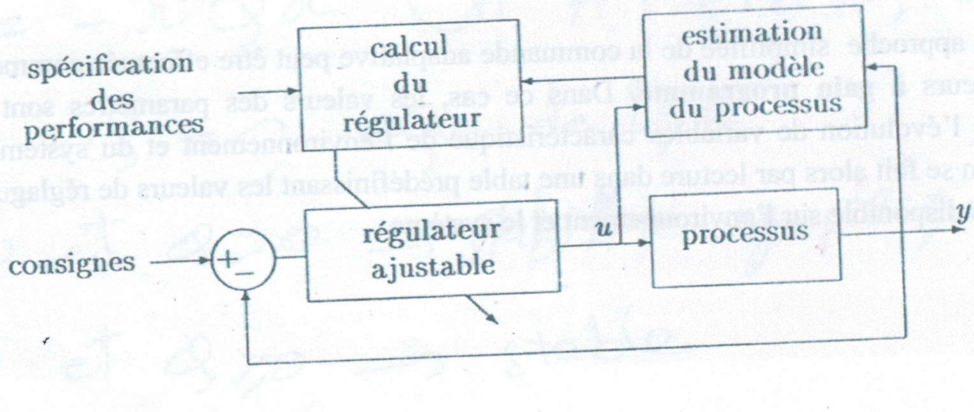
2.3. Approches de la commande adaptative

Deux approches principales existent pour la commande adaptative des systèmes à paramètres inconnues ou variable dans le temps.

❖ La **commande adaptative directe,** dans laquelle les paramètres du régulateur sont ajustés directement et en temps réel à partir de comparaison entre performances réel et les performances désirées (c'est le cas en particulier de la commande adaptative à modèle de référence).



❖ La **commande adaptative indirecte**, qui suppose une estimation des paramètres du système par une procédure d'identification (c'est le cas de régulateur auto-ajustables). Ce dernier type de commande adaptative qui tient compte les caractéristiques d'évolution du système, est en fait plus utilisé que le précédent.



Adaptive control of linear systems

The principle of feedback control is to maintain a consistent performance when there are uncertainties in the system or changes in the setpoints through a feedback controller using the measurements of the system performance, mainly the outputs. Many controllers are with fixed controller parameters, such as the controllers designed by normal state feedback control, and H_∞ control methods. The basic aim of adaptive control also is to maintain a consistent performance of a system in the presence of uncertainty or unknown variation in plant parameters, but with changes in the controller parameters, adapting to the changes in the performance of the control system. Hence, there is an adaptation in the controller setting subject to the performance of the closed-loop system. How the controller parameters change is decided by the adaptive laws, which are often designed based on the stability analysis of the adaptive control system.

A number of design methods have been developed for adaptive control. Model Reference Adaptive Control (MRAC) consists of a reference model which produces the desired output, and the difference between the plant output and the reference output is then used to adjust the control parameters and the control input directly. MRAC is often in continuous-time domain, and for deterministic plants. Self-Tuning Control (STC) estimates system parameters and then computes the control input from the estimated parameters. STC is often in discrete-time and for stochastic plants. Furthermore, STC often has a separate identification procedure for estimation of the system parameters, and is referred to as indirect adaptive control, while MRAC adapts to the changes in the controller parameters, and is referred to as direct adaptive control. In general, the stability analysis of direct adaptive control is less involved than that of indirect adaptive control, and can often be carried out using Lyapunov functions. In this chapter, we focus on the basic design method of MRAC.

Compared with the conventional control design, adaptive control is more involved, with the need to design the adaptation law. MRAC design usually involves the following three steps:

- Choose a control law containing variable parameters.
- Design an adaptation law for adjusting those parameters.
- Analyse the stability properties of the resulting control system.

7.1 MRAC of first-order systems

The basic design idea can be clearly demonstrated by first-order systems. Consider a first-order system

$$\dot{y} + a_p y = b_p u, \tag{7.1}$$

where y and $u \in \mathbb{R}$ are the system output and input respectively, and a_p and b_p are unknown constant parameters with $\mathrm{sgn}(b_p)$ known. The output y is to follow the output of the reference model

$$\dot{y}_m + a_m y_m = b_m r. \tag{7.2}$$

The reference model is stable, i.e., $a_m > 0$. The signal r is the reference input. The design objective is to make the tracking error $e = y - y_m$ converge to 0.

Let us first design a Model Reference Control (MRC), that is, the control design assuming all the parameters are known, to ensure that the output y follows y_m. Rearrange the system model as

$$\dot{y} + a_m y = b_p \left(u - \frac{a_p - a_m}{b_p} y \right)$$

and therefore we obtain

$$\begin{aligned} \dot{e} + a_m e &= b_p \left(u - \frac{a_p - a_m}{b_p} y - \frac{b_m}{b_p} r \right) \\ &:= b_p \left(u - a_u y - a_r r \right), \end{aligned}$$

where

$$\begin{aligned} a_y &= \frac{a_p - a_m}{b_p}, \\ a_r &= \frac{b_m}{b_p}. \end{aligned}$$

If all the parameters are known, the control law is designed as

$$u = a_r r + a_y y \tag{7.3}$$

and the resultant closed-loop system is given by

$$\dot{e} + a_m e = 0.$$

The tracking error converges to zero exponentially.

One important design principle in adaptive control is the so-called *the certainty equivalence principle*, which suggests that the unknown parameters in the control design are replaced by their estimates. Hence, when the parameters are unknown, let $\hat{a}_r$ and $\hat{a}_y$ denote their estimates of a_r and a_y, and the control law, based on the certainty equivalence principle, is given by

$$u = \hat{a}_r r + \hat{a}_y y. \tag{7.4}$$

Note that the parameters a_r and a_y are the parameters of the controllers, and they are related to the original system parameters a_p and b_p, but not the original system parameters themselves.

The certainty equivalence principle only suggests a way to design the adaptive control input, not how to update the parameter estimates. Stability issues must be considered when deciding the adaptive laws, i.e., the way how estimated parameters are updated. For first-order systems, the adaptive laws can be decided from Lyapunov function analysis.

With the proposed adaptive control input (7.4), the closed-loop system dynamics are described by

$$\dot{e} + a_m e = b_p(-\tilde{a}_y y - \tilde{a}_r r), \tag{7.5}$$

where $\tilde{a}_r = a_r - \hat{a}_r$ and $\tilde{a}_y = a_y - \hat{a}_y$. Consider the Lyapunov function candidate

$$V = \frac{1}{2}e^2 + \frac{|b_p|}{2\gamma_r}\tilde{a}_r^2 + \frac{|b_p|}{2\gamma_y}\tilde{a}_y^2, \tag{7.6}$$

where γ_r and γ_y are constant positive real design parameters. Its derivative along the trajectory (7.5) is given by

$$\dot{V} = -a_m e^2 + \tilde{a}_r\left(|b_p|\frac{\dot{\tilde{a}}_r}{\gamma_r} - eb_p r\right) + \tilde{a}_y\left(|b_p|\frac{\dot{\tilde{a}}_y}{\gamma_y} - eb_p y\right).$$

If we can set

$$|b_p|\frac{\dot{\tilde{a}}_r}{\gamma_r} - eb_p r = 0, \tag{7.7}$$

$$|b_p|\frac{\dot{\tilde{a}}_y}{\gamma_y} - eb_p y = 0, \tag{7.8}$$

we have

$$\dot{V} = -a_m e^2. \tag{7.9}$$

Noting that $\dot{\hat{a}}_r = -\dot{\tilde{a}}_r$ and $\dot{\hat{a}}_y = -\dot{\tilde{a}}_y$, the conditions in (7.7) and (7.8) can be satisfied by setting the adaptive laws as

$$\dot{\hat{a}}_r = -\text{sgn}(b_p)\gamma_r e r, \tag{7.10}$$

$$\dot{\hat{a}}_y = -\text{sgn}(b_p)\gamma_y e y. \tag{7.11}$$

The positive real design parameters γ_r and γ_y are often referred to as adaptive gains, as they can affect the speed of parameter adaptation.

From (7.9) and Theorem 4.2, we conclude that the system is Lyapunov stable with all the variables e, $\tilde{a}_r$ and $\tilde{a}_y$ bounded, and hence the boundedness of $\hat{a}_r$ and $\hat{a}_y$.

However, based on the stability theorems introduced in Chapter 4, we cannot conclude anything about the tracking error e other than its boundedness. In order to do it, we need to introduce an important lemma for stability analysis of adaptive control systems.

Lemma 7.1 (Barbalat's lemma). *If a function* $f(t) : \mathbb{R} \longrightarrow \mathbb{R}$ *is uniformly continuous for* $t \in [0,\infty)$*, and* $\int_0^\infty f(t)dt$ *exists, then* $\lim_{t\to\infty} f(t) = 0$.

From (7.9), we can show that

$$\int_0^\infty e^2(t)dt = \frac{V(0) - V(\infty)}{a_m} < \infty. \tag{7.12}$$

Therefore, we have established that $e \in L_2 \cap L_\infty$ and $\dot{e} \in L_\infty$. Since $\dot{e}$ and e are bounded, e^2 is uniformly continuous. Therefore, we can conclude from Barbalat's lemma that $\lim_{t\to\infty} e^2(t) = 0$, and hence $\lim_{t\to\infty} e(t) = 0$.

We summarise the stability result in the following lemma.

Lemma 7.2. *For the first-order system (7.1) and the reference model (7.2), the adaptive control input (7.4) together with the adaptive laws (7.10) and (7.11) ensures the boundedness of all the variables in the closed-loop system, and the convergence to zero of the tracking error.*

Remark 7.1. The stability analysis ensures the convergence to zero of the tracking error, but nothing can be told about the convergence of the estimated parameters. The estimated parameters are assured to be bounded from the stability analysis. In general, the convergence of the tracking error to zero and the boundedness of the adaptive parameters are stability results that we can establish for MRAC. The convergence of the estimated parameters may be achieved by imposing certain conditions of the reference signal to ensure the system is excited enough. This is similar to the concept of persistent excitation for system identification. ◁

Example 7.1. Consider a first-order system

$$G_p = \frac{b}{s+a},$$

where $b = 1$ and a is an unknown constant parameter. We will design an adaptive controller such that the output of the system follows the output of the reference model

$$G_m = \frac{1}{s+2}.$$

We can directly use the result presented in Lemma 7.2, i.e., we use the adaptive laws (7.10) and (7.11) and the control input (7.4). Since b is known, we only have one unknown parameter, and it is possible to design a simpler control based on the same design principle.

From the system model, we have

$$\dot{y} + ay = u,$$

which can be changed to

$$\dot{y} + 2y = u - (a-2)y.$$

Subtracting the reference model

$$\dot{y}_m + 2y_m = r,$$

we obtain that

$$\dot{e} + 2e = u - a_y y - r.$$

where $a_y = a - 2$. We then design the adaptive law and control input as

$$\begin{aligned} \dot{\hat{a}}_y &= -\gamma_y e y, \\ u &= \hat{a}_y y + r. \end{aligned}$$

The stability analysis follows the same discussion that leads to Lemma 7.2. Simulation study has been carried out with $a = -1$, $\gamma = 10$ and $r = 1$. The simulation results are shown in Figure 7.1. The figure shows that the estimated parameter converges to the true value $a_y = -3$. The convergence of the estimated parameters is not guaranteed by Lemma 7.2. Indeed, some strong conditions on the input or reference signal are needed to generate enough excitation for the parameter estimation to achieve the convergence of the estimated parameters in general. ◁

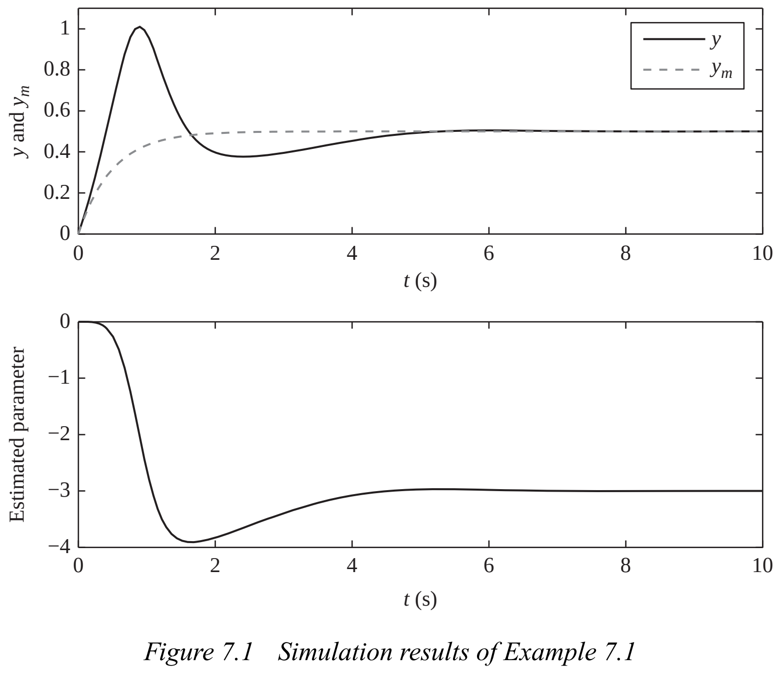


Figure 7.1 Simulation results of Example 7.1

7.2 Model reference control

It is clear from MRAC design for first-order systems that an MRC input is designed first which contains unknown parameters, and the adaptive control input is then obtained based on the certainty equivalence principle. Hence, MRC design is the first step for MRAC. Furthermore, MRC itself deserves a brief introduction, as it is different from the classical control design methods shown in standard undergraduate texts. In this section, we will start with MRC for systems with relative degree 1, and then move on to MRC of systems with high-order relative degrees.

Consider an nth-order system with the transfer function

$$y(s) = k_p \frac{Z_p(s)}{R_p(s)} u(s), \tag{7.13}$$

where $y(s)$ and $u(s)$ denote the system output and input in frequency domain; k_p is the high frequency gain; and Z_p and R_p are monic polynomials with orders of $n - \rho$ and n respectively with ρ as the relative degree. The reference model is chosen to have the same relative degree of the system, and is described by

$$y_m(s) = k_m \frac{Z_m(s)}{R_m(s)} r(s), \tag{7.14}$$

where $y_m(s)$ is the reference output for $y(s)$ to follow; $r(s)$ is a reference input; and $k_m > 0$ and Z_m and R_m are monic Hurwitz polynomials.

Remark 7.2. A monic polynomial is a polynomial whose leading coefficient, the coefficient of the highest power, is 1. A polynomial is said to be Hurwitz if all its roots are with negative real parts, i.e., its roots locate in the open left half of the complex plane. The high-frequency gain is the leading coefficient of the numerator of a transfer function. ◁

The objective of MRC is to design a control input u such that the output of the system asymptotically follows the output of the reference model, i.e., $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$.

Note that in this chapter, we abuse the notations of y, u and r by using same notations for the functions in time domain and their Laplace transformed functions in the frequency domain. It should be clear from the notations that $y(s)$ is the Laplace transform of $y(t)$ and similarly for u and r.

To design MRC for systems with $\rho = 1$, we follow a similar manipulation to the first-order system by manipulating the transfer functions. We start with

$$y(s)R_p(s) = k_p Z_p(s) u(s)$$

and then

$$y(s)R_m(s) = k_p Z_p(s) u(s) - (R_p(s) - R_m(s)) y(s).$$

Note that $R_p(s) - R_m(s)$ is a polynomial with order $n-1$, and $\frac{R_m(s)-R_p(s)}{Z_m(s)}$ is a proper transfer function, as $R_p(s)$ and $R_m(s)$ are monic polynomials. Hence, we can write

$$y(s)R_m(s) = k_p Z_m(s)\left(\frac{Z_p(s)}{Z_m(s)}u(s) + \frac{R_m(s)-R_p(s)}{Z_m(s)}y(s)\right).$$

If we parameterise the transfer functions as

$$\begin{aligned}\frac{Z_p(s)}{Z_m(s)} &= 1 - \frac{\theta_1^T\alpha(s)}{Z_m(s)},\\ \frac{R_m(s)-R_p(s)}{Z_m(s)}y(s) &= -\frac{\theta_2^T\alpha(s)}{Z_m(s)}y(s) - \theta_3,\end{aligned}$$

where $\theta_1 \in \mathbb{R}^{n-1}$, $\theta_2 \in \mathbb{R}^{n-1}$ and $\theta_3 \in \mathbb{R}$ are constants and

$$\alpha(s) = [s^{n-2}, \ldots, 1]^T,$$

we obtain that

$$y(s) = k_p\frac{Z_m(s)}{R_m(s)}\left(u(s) - \frac{\theta_1^T\alpha(s)}{Z_m(s)}u(s) - \frac{\theta_2^T\alpha(s)}{Z_m(s)}y(s) - \theta_3 y(s)\right). \tag{7.15}$$

Hence, we have the dynamics of tracking error given by

$$e_1(s) = k_p\frac{Z_m(s)}{R_m(s)}\left(u(s) - \frac{\theta_1^T\alpha(s)}{Z_m(s)}u(s) - \frac{\theta_2^T\alpha(s)}{Z_m(s)}y(s) - \theta_3 y(s) - \theta_4 r\right), \tag{7.16}$$

where $e_1 = y - y_m$ and $\theta_4 = \frac{k_m}{k_p}$.

The control input for MRC is given by

$$\begin{aligned}u(s) &= \frac{\theta_1^T\alpha(s)}{Z_m(s)}u(s) + \frac{\theta_2^T\alpha(s)}{Z_m(s)}y + \theta_3 y + \theta_4 r(s)\\ &:= \theta^T\omega,\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}\theta^T &= [\theta_1^T, \theta_4^T, \theta_3, \theta_4],\\ \omega &= [\omega_1^T, \omega_2^T, y, r]^T,\end{aligned}$$

with

$$\begin{aligned}\omega_1 &= \frac{\alpha(s)}{Z_m(s)}u,\\ \omega_2 &= \frac{\alpha(s)}{Z_m(s)}y.\end{aligned}$$

Remark 7.3. The control design shown in (7.17) is a dynamic feedback controller. Each element in the transfer matrix $\frac{\alpha(s)}{Z_m(s)}$ is strictly proper, i.e., with relative degree greater than or equal to 1. The total number of parameters in θ equals $2n$. ◁

Lemma 7.3. *For the system (7.13) with relative degree 1, the control input (7.17) solves MRC problem with the reference model (7.14) and* $\lim_{t\to\infty}(y(t) - y_m(t)) = 0$.

Proof. With the control input (7.17), the closed-loop dynamics are given by

$$e_1(s) = k_p \frac{Z_m(s)}{R_m(s)} \epsilon(s),$$

where $\epsilon(s)$ denotes exponentially convergent signals due to non-zero initial values. The reference model is stable, and then the track error $e_1(t)$ converges to zero exponentially. □

Example 7.2. Design MRC for the system

$$y(s) = \frac{s+1}{s^2 - 2s + 1} u(s)$$

with the reference model

$$y_m(s) = \frac{s+3}{s^2 + 2s + 3} r(s).$$

We follow the procedures shown early to obtain the MRC control. From the transfer function of the system, we have

$$y(s)(s^2 + 2s + 3) = (s+1)u(s) + (4s+2)y(s),$$

which leads to

$$\begin{aligned} y(s) &= \frac{s+3}{s^2+2s+3}\left(\frac{s+1}{s+3}u(s) + \frac{4s+2}{s+3}y(s)\right) \\ &= \frac{s+3}{s^2+2s+3}\left(u(s) - \frac{2}{s+3}u(s) - \frac{10}{s+3}y(s) + 4y(s)\right). \end{aligned}$$

Subtracting it by the reference model, we have

$$e_1(s) = \frac{s+3}{s^2+2s+3}\left(u(s) - \frac{2}{s+3}u(s) - \frac{10}{s+3}y(s) + 4y(s) - r(s)\right),$$

which leads to the MRC control input

$$\begin{aligned} u(s) &= \frac{2}{s+3}u(s) + \frac{10}{s+3}y(s) - 4y(s) + r(s) \\ &= [2\ 10\ -4\ 1][\omega_1(s)\ \omega_2(s)\ y(s)\ r(s)]^T, \end{aligned}$$

where

$$\omega_1(s) = \frac{1}{s+3}u(s),$$
$$\omega_2(s) = \frac{1}{s+3}y(s).$$

Note that the control input in the time domain is given by

$$u(s) = [2\ 10\ \ -4\ 1][\omega_1(t)\ \omega_2(t)\ y(t)\ r(t)]^T,$$

where

$$\dot{\omega}_1 = -3\omega_1 + u,$$
$$\dot{\omega}_2 = -3\omega_2 + y.$$

◁

For a system with $\rho > 1$, the input in the same format as (7.17) can be obtained. The only difference is that Z_m is of order $n - \rho < n - 1$. In this case, we let $P(s)$ be a monic and Hurwitz polynomial with order $\rho - 1$ so that $Z_m(s)P(s)$ is of order $n - 1$. We adopt a slightly different approach from the case of $\rho = 1$.

Consider the identity

$$\begin{aligned} y(s) &= \frac{Z_m(s)}{R_m(s)}\left(\frac{R_m(s)P(s)}{Z_m(s)P(s)}y(s)\right) \\ &= \frac{Z_m(s)}{R_m(s)}\left(\frac{Q(s)R_p(s) + \Delta(s)}{Z_m(s)P(s)}y(s)\right). \end{aligned} \tag{7.18}$$

Note that the second equation in (7.18) follows from the identity

$$R_m(s)P(s) = Q(s)R_p(s) + \Delta(s),$$

where $Q(s)$ is a monic polynomial with order $n - \rho - 1$, and $\Delta(s)$ is a polynomial with order $n - 1$. In fact $Q(s)$ can be obtained by dividing $R_m(s)P(s)$ by $R_p(s)$ using long division, and $\Delta(s)$ is the remainder of the polynomial division. From the transfer function of the system, we have

$$R_p(s)y(s) = k_p Z_p(s)u(s).$$

Substituting it into (7.18), we have

$$y(s) = k_p\frac{Z_m(s)}{R_m(s)}\left(\frac{Q(s)Z_p(s)}{Z_m(s)P(s)}u + \frac{k_p^{-1}\Delta(s)}{Z_m(s)P(s)}y(s)\right).$$

Similar to the case for $\rho = 1$, if we parameterise the transfer functions as

$$\frac{Q(s)Z_p(s)}{Z_m(s)P(s)} = 1 - \frac{\theta_1^T\alpha(s)}{Z_m(s)P(s)},$$
$$\frac{k_p^{-1}\Delta}{Z_m(s)P(s)} = -\frac{\theta_2^T\alpha(s)}{Z_m(s)P(s)} - \theta_3$$

where $\theta_1 \in \mathbb{R}^{n-1}$ and $\theta_2 \in \mathbb{R}^{n-1}$ and $\theta_3 \in \mathbb{R}$ are constants and

$$\alpha(s) = [s^{n-2}, \ldots, 1]^T,$$

we obtain that

$$y(s) = k_p \frac{Z_m(s)}{R_m(s)} \left(u(s) - \frac{\theta_1^T \alpha(s)}{Z_m(s)P(s)} u(s) - \frac{\theta_2^T \alpha(s)}{Z_m(s)P(s)} y(s) - \theta_3 y(s) \right).$$

Hence, we have the dynamics of tracking error given by

$$e_1(s) = k_p \frac{Z_m(s)}{R_m(s)} \left(u(s) - \frac{\theta_1^T \alpha(s)}{Z_m(s)P(s)} u(s) - \frac{\theta_2^T \alpha(s)}{Z_m(s)P(s)} y(s) - \theta_3 y(s) - \theta_4 r \right),$$

where $e_1 = y - y_m$ and $\theta_4 = \frac{k_m}{k_p}$. The control input is designed as

$$\begin{aligned} u &= \frac{\theta_1^T \alpha(s)}{Z_m(s)P(s)} u + \frac{\theta_2^T \alpha(s)}{Z_m(s)P(s)} y + \theta_3 y + \theta_4 r \\ &:= \theta^T \omega \end{aligned} \tag{7.19}$$

with the same format as (7.17) except

$$\begin{aligned} \omega_1 &= \frac{\alpha(s)}{Z_m(s)P(s)} u, \\ \omega_2 &= \frac{\alpha(s)}{Z_m(s)P(s)} y. \end{aligned}$$

Remark 7.4. The final control input is in the same format as shown for the case $\rho = 1$. The filters for w_1 and w_2 are in the same order as in the case for $\rho = 1$, as the order of $Z_m(s)P(s)$ is still $n - 1$. ◁

Lemma 7.4. *For the system (7.13) with relative degree $\rho > 1$, the control input (7.19) solves MRC problem with the reference model (7.14) and* $\lim_{t\to\infty} (y(t) - y_m(t)) = 0$.

The proof is the same as the proof for Lemma 7.3.

Example 7.3. Design MRC for the system

$$y(s) = \frac{1}{s^2 - 2s + 1} u$$

with the reference model

$$y_m(s) = \frac{1}{s^2 + 2s + 3} r.$$

The relative degree of the system is 2. We set $P = s + 1$. Note that

$$(s^2 + 2s + 3)(s + 1) = (s + 5)(s^2 - 2s + 1) + (14s - 2).$$

From the reference model, we have

$$
\begin{aligned}
y(s) &= \frac{1}{s^2+2s+3}\left(\frac{(s^2+2s+3)(s+1)}{s+1}y(s)\right) \\
&= \frac{1}{s^2+2s+3}\left(\frac{(s+5)(s^2-2s+1)y(s)+(14s-2)y(s)}{s+1}\right) \\
&= \frac{1}{s^2+2s+3}\left(\frac{(s+5)u(s)+(14s-2)y(s)}{s+1}\right) \\
&= \frac{1}{s^2+2s+3}\left(u(s)+\frac{4}{s+1}u(s)-16\frac{1}{s+1}y(s)+14y(s)\right).
\end{aligned}
$$

The dynamics of the tacking error are given by

$$
e_1(s) = \frac{1}{s^2+2s+3}\left(u(s)+\frac{4}{s+1}u(s)-16\frac{1}{s+1}y(s)+14y(s)-r(s)\right).
$$

We can then design the control input as

$$
u = [-1,\ -16,\ 14,\ 1][\omega_1,\ \omega_2,\ y,\ r]^T
$$

with

$$
\begin{aligned}
\omega_1 &= \frac{1}{s+1}u, \\
\omega_2 &= \frac{1}{s+1}y.
\end{aligned}
$$

◁

7.3 MRAC of linear systems with relative degree 1

Adaptive control deals with uncertainties in terms of unknown constant parameters. It may be used to tackle some changes or variations in model parameters in adaptive control application, but the stability analysis will be carried under the assumption the parameters are constants. There are other common assumptions for adaptive control which are listed below:

- the known system order n
- the known relative degree ρ
- the minimum phase of the plant
- the known sign of the high frequency gain $\mathrm{sgn}(k_p)$

In this section, we present MRAC design for linear systems with relative degree 1.

Consider an nth-order system with the transfer function

$$
y(s) = k_p\frac{Z_p(s)}{R_p(s)}u(s), \tag{7.20}
$$

where $y(s)$ and $u(s)$ denote the system output and input in frequency domain; k_p is the high frequency gain; and Z_p and R_p are monic polynomials with orders of $n-1$ and n respectively. This system is assumed to be minimum phase, i.e., $Z_p(s)$ is a Hurwitz polynomial, and the sign of the high-frequency gain, $\text{sgn}(k_p)$, is known. The coefficients of the polynomials and the value of k_p are constants and unknown. The reference model is chosen to have the relative degree 1 and strictly positive real, and is described by

$$y_m(s) = k_m \frac{Z_m(s)}{R_m(s)} r(s), \tag{7.21}$$

where $y_m(s)$ is the reference output for $y(s)$ to follow, $r(s)$ is a reference input, and $Z_m(s)$ and $R_m(s)$ are monic polynomials and $k_m > 0$. Since the reference model is strictly positive real, Z_m and R_m are Hurwitz polynomials.

MRC shown in the previous section gives the control design in (7.17). Based on the certainty equivalence principle, we design the adaptive control input as

$$u(s) = \hat{\theta}^T \omega, \tag{7.22}$$

where $\hat{\theta}$ is an estimate of the unknown vector $\theta \in \mathbb{R}^{2n}$, and ω is given by

$$\omega = [\omega_1^T, \omega_2^T, y, r]^T$$

with

$$\begin{aligned} \omega_1 &= \frac{\alpha(s)}{Z_m(s)} u, \\ \omega_2 &= \frac{\alpha(s)}{Z_m(s)} y. \end{aligned}$$

With the designed adaptive control input, it can be obtained, from the tracking error dynamics shown in (7.16), that

$$\begin{aligned} e_1(s) &= k_p \frac{Z_m(s)}{R_m(s)} (\hat{\theta}^T \omega - \theta^T \omega) \\ &= k_m \frac{Z_m(s)}{R_m(s)} \left(-\frac{k_p}{k_m} \tilde{\theta}^T \omega \right) \end{aligned} \tag{7.23}$$

where $\tilde{\theta} = \theta - \hat{\theta}$.

To analyse the stability using a Lyapunov function, we put the error dynamics in the state space form as

$$\begin{aligned} \dot{e} &= A_m e + b_m \left(-\tfrac{k_p}{k_m} \tilde{\theta}^T \omega \right) \\ e_1 &= c_m^T e \end{aligned} \tag{7.24}$$

where (A_m, b_m, c_m) is a minimum state space realisation of $k_m \frac{Z_m(s)}{R_m(s)}$, i.e.,

$$c_m^T (sI - A_m)^{-1} b_m = k_m \frac{Z_m(s)}{R_m(s)}.$$

Since (A_m, b_m, c_m) is a strictly positive real system, from Kalman–Yakubovich lemma (Lemma 5.4), there exist positive definite matrices P and Q such that

$$A_m^T P_m + P_m A_m = -Q_m, \tag{7.25}$$

$$P_m b_m = c_m. \tag{7.26}$$

Define a Lyapunov function candidate as

$$V = \frac{1}{2} e^T P_m e + \frac{1}{2} \left| \frac{k_p}{k_m} \right| \tilde{\theta}^T \Gamma^{-1} \tilde{\theta},$$

where $\Gamma \in \mathbb{R}^{2n}$ is a positive definite matrix. Its derivative is given by

$$\dot{V} = \frac{1}{2} e^T (A_m^T P_m + P_m A_m) e + e^T P_m b_m \left(-\frac{k_p}{k_m} \tilde{\theta}^T \omega \right) + \left| \frac{k_p}{k_m} \right| \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}}$$

Using the results from (7.25) and (7.26), we have

$$\begin{aligned} \dot{V} &= -\frac{1}{2} e^T Q_m e + e_1 \left(-\frac{k_p}{k_m} \tilde{\theta}^T \omega \right) + \left| \frac{k_p}{k_m} \right| \tilde{\theta}^T \Gamma^{-1} \dot{\tilde{\theta}} \\ &= -\frac{1}{2} e^T Q_m e + \left| \frac{k_p}{k_m} \right| \tilde{\theta}^T \left(\Gamma^{-1} \dot{\tilde{\theta}} - \operatorname{sgn}(k_p) e_1 \omega \right). \end{aligned}$$

Hence, the adaptive law is designed as

$$\dot{\hat{\theta}} = -\operatorname{sgn}(k_p) \Gamma e_1 \omega, \tag{7.27}$$

which results in

$$\dot{V} = -\frac{1}{2} e^T Q_m e.$$

We can now conclude the boundedness of e and $\hat{\theta}$. Furthermore it can be shown that $e \in L_2$ and $\dot{e}_1 \in L_\infty$. Therefore, from Barbalat's lemma we have $\lim_{t \to \infty} e_1(t) = 0$. The boundedness of other system state variables can be established from the minimum-phase property of the system.

We summarise the stability analysis for MRAC of linear systems with relative degree 1 in the following theorem.

Theorem 7.5. *For the first-order system (7.20) and the reference model (7.21), the adaptive control input (7.22) together with the adaptive law (7.27) ensures the boundedness of all the variables in the closed-loop system, and the convergence to zero of the tracking error.*

Remark 7.5. The stability result shown in Theorem 7.5 only guarantees the convergence of the tracking error to zero, not the convergence of the estimated parameters. In the stability analysis, we use Kalman–Yakubovich lemma for the definition of Lyapunov function and the stability proof. That is why we choose the reference model to be strictly positive real. From the control design point of view, we do not need to know the actual values of P_m and Q_m, as long as they exist, which is guaranteed by the selection of a strictly positive real model. Also it is clear from the stability analysis, that the unknown parameters must be constant. Otherwise, we would not have $\dot{\hat{\theta}} = -\dot{\tilde{\theta}}$. ◁

7.4 MRAC of linear systems with high relatives

In this section, we will introduce adaptive control design for linear systems with their relative degrees higher than 1. Similar to the case for relative degree 1, the certainty equivalence principle can be applied to the control design, but the designs of the adaptive laws and the stability analysis are much more involved, due to the higher relative degrees. One difficulty is that there is not a clear choice of Lyapunov function candidate as in the case of $\rho = 1$.

Consider an nth-order system with the transfer function

$$y(s) = k_p \frac{Z_p(s)}{R_p(s)} u(s), \tag{7.28}$$

where $y(s)$ and $u(s)$ denote the system output and input in frequency domain, k_p is the high frequency gain, Z_p and R_p are monic polynomials with orders of $n - \rho$ and n respectively, with $\rho > 1$ being the relative degree of the system. This system is assumed to be minimum phase, i.e., $Z_p(s)$ is Hurwitz polynomial, and the sign of the high-frequency gain, $\mathrm{sgn}(k_p)$, is known. The coefficients of the polynomials and the value of k_p are constants and unknown. The reference model is chosen as

$$y_m(s) = k_m \frac{Z_m(s)}{R_m(s)} r(s) \tag{7.29}$$

where $y_m(s)$ is the reference output for $y(s)$ to follow; $r(s)$ is a reference input; and $Z_m(s)$ and $R_m(s)$ are monic polynomials with orders $n - \rho$ and n respectively and $k_m > 0$. The reference model (7.29) is required to satisfy an additional condition that there exists a monic and Hurwitz polynomial $P(s)$ of order $n - \rho - 1$ such that

$$y_m(s) = k_m \frac{Z_m(s)P(s)}{R_m(s)} r(s) \tag{7.30}$$

is strictly positive real. This condition also implies that Z_m and R_m are Hurwitz polynomials.

MRC shown in the previous section gives the control design in (7.19). We design the adaptive control input, again using the certainty equivalence principle, as

$$u = \hat{\theta}^T \omega, \tag{7.31}$$

where $\hat{\theta}$ is an estimate of the unknown vector $\theta \in \mathbb{R}^{2n}$, and ω is given by

$$\omega = [\omega_1^T, \omega_2^T, y, r]^T$$

with

$$\omega_1 = \frac{\alpha(s)}{Z_m(s)P(s)}u,$$

$$\omega_2 = \frac{\alpha(s)}{Z_m(s)P(s)}y.$$

The design of adaptive law is more involved, and we need to examine the dynamics of the tacking error, which are given by

$$\begin{aligned} e_1 &= k_p \frac{Z_m}{R_m}(u - \theta^T \phi) \\ &= k_m \frac{Z_m P(s)}{R_m} \left(k(u_f - \theta^T \phi) \right), \end{aligned} \tag{7.32}$$

where

$$k = \frac{k_p}{k_m}, \quad u_f = \frac{1}{P(s)}u \quad \text{and} \quad \phi = \frac{1}{P(s)}\omega.$$

An auxiliary error is constructed as

$$\epsilon = e_1 - k_m \frac{Z_m P(s)}{R_m} \left(\hat{k}(u_f - \hat{\theta}^T \phi) \right) - k_m \frac{Z_m P(s)}{R_m} \left(\epsilon n_s^2 \right), \tag{7.33}$$

where $\hat{k}$ is an estimate of k, $n_s^2 = \phi^T \phi + u_f^2$. The adaptive laws are designed as

$$\dot{\hat{\theta}} = -\text{sgn}(b_p)\Gamma \epsilon \phi, \tag{7.34}$$

$$\dot{\hat{k}} = \gamma \epsilon (u_f - \hat{\theta}^T \phi). \tag{7.35}$$

With these adaptive laws, a stability result can be obtained for the boundedness of parameter estimates and the convergence of the tracking error. For the completeness, we state the theorem below without giving the proof.

Theorem 7.6. *For the system (7.28) and the reference model (7.29), the adaptive control input (7.31) together with the adaptive laws (7.34) and (7.35) ensures the boundedness of all the variables in the closed-loop system, and the convergence to zero of the tracking error.*