Solutions of tutorial exercises (1,2,3,4) set 2 :

Hocine RANDJI

November 25, 2023

This document is supplemented for the second chapter lecture notes (Analyses 1).

Exercise 01:

Let us take $n \in \mathbb{N}$, then we have:

(a) For $n \in \mathbb{N}$ we have $-\frac{1}{2}, \frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{7}{14}$.

(b) For $n \in \mathbb{N}^*$ we have $2, 0, \frac{2}{9}, 0, \frac{2}{25}$.

(c) For $n \in \mathbb{N}^*$ we have $\frac{1}{2}, -\frac{1}{8}, \frac{1}{48}, -\frac{1}{384}, \frac{1}{3840}$.

(d) For $n \in \mathbb{N}^*$ we have $\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}, \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}$

Exercise 02:

(a) For values of $u_1 = 0.22222\ldots$, $u_5 = 0.56000\ldots$, $u_{10} = 0.64444\ldots$, $u_{100} = 0.73827\ldots$, $u_{1000} = 0.74881\ldots$, $u_{10000} = 0.74988\ldots$ and $u_{100000} = 0.74998\ldots$. A reasonable guess is that the limit is $3/4$. It's important to note that this limit becomes evident only for sufficiently large values of n.

(b)To verify this limit by using the definition: We need to demonstrate that for any given positive value ε, there exists a corresponding number N (dependent on ε) such that $|u_n - \frac{3}{4}| < \varepsilon$ for all $n > N$. By manipulating the expression, $\left|\frac{3n-1}{4n+5} - \frac{3}{4}\right| < \varepsilon$, we find that $\left|-\frac{19}{4(4n+5)}\right| < \varepsilon$, which leads to $\frac{19}{4(4n+5)} < \varepsilon \iff \frac{4(4n+5)}{19} > \frac{1}{\varepsilon}$.

We can simplify and derive conditions such as $4n+5 > \frac{19}{4\varepsilon} \iff n > \frac{1}{4}\left(\frac{19}{4\varepsilon} - 5\right)$. Choosing $N = \left[\frac{1}{4}\left(\frac{19}{4\varepsilon} - 5\right)\right] + 1$ accordingly ensures that the limit as n approaches infinity is indeed $3/4$.

Exercise 03:

(1) $\lim\limits_{n \to +\infty} \frac{3n-1}{2n+3} = \frac{3}{2} \iff \forall \varepsilon > 0, \exists ? n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_\varepsilon \Rightarrow \left|\frac{3n-1}{2n+3} - \frac{3}{2}\right| < \varepsilon$

We use $\left|\frac{3n-1}{2n+3} - \frac{3}{2}\right| < \varepsilon$, then we have $\left|\frac{2(3n-1)-3(2n+3)}{2(2n+3)}\right| = \left|\frac{6n-2-6n-9}{(4n+6)}\right| = \frac{11}{(4n+6)} < \varepsilon \iff \frac{11}{4\varepsilon} - \frac{3}{2} < n$.

For this, it is sufficient to take $n_\varepsilon = \lceil \frac{11}{4\varepsilon} - \frac{3}{2} \rceil + 1$.

(2) $\lim\limits_{n \to +\infty} \frac{(-1)^n}{2n} = 0 \iff \forall \varepsilon > 0, \exists ? n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_\varepsilon \Rightarrow \left|\frac{1}{2^n}\right| < \varepsilon$. We have $\frac{1}{2^n} < \varepsilon$ leads to $-\frac{\ln \varepsilon}{\ln 2} < n$.

For this, it is sufficient to take $n_\varepsilon = \lceil -\ln(\varepsilon)/\ln(2) \rceil + 1$.

(3) $\lim\limits_{n \to +\infty} \frac{2\ln(1+n)}{\ln(n)} = 2 \iff \forall \varepsilon > 0, \exists ? n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_\varepsilon \Rightarrow \left|\frac{2\ln(1+n)}{\ln(n)} - 2\right| < \varepsilon$.

So we take, $\left|\frac{2\ln(1+n)}{\ln(n)} - 2\right| = \left|\frac{2\ln(1+n) - 2\ln(n)}{\ln(n)}\right| = 2\left|\frac{\ln(\frac{1+n}{n})}{\ln(n)}\right| = \frac{2\ln\left(\frac{1}{n}+1\right)}{\ln n}$

Then we can use: $\forall n \in \mathbb{N}^* : \frac{1}{n} \leq 1$ so that we have $\frac{1}{n} + 1 \leq 2$, which leads to $\frac{2\ln(\frac{1}{n}+1)}{\ln n} \leq \frac{2\ln 2}{\ln n}$. Thus, we can choose $\left|\frac{2\ln(1+n)}{\ln(n)} - 2\right| < \frac{2\ln(\frac{1}{n}+1)}{\ln n} < \varepsilon$, which leads to $n > e^{\frac{2\ln 2}{\ln n}}$. For this, it is sufficient to take $n_\varepsilon = \lceil e^{2\ln(2)/\varepsilon} \rceil + 1$.

(4) $\lim_{n\to+\infty} 3^n = +\infty \iff (\forall A > 0, \exists ? n_A \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_A \Rightarrow 3^n > A)$. We have $3^n > A \iff n > \frac{\ln A}{\ln 3}$. For this, take $n_A = \lceil |\ln(A)/\ln(3)| \rceil + 1$.

(5) $\lim_{n\to+\infty} \frac{-5n^2-2}{4n} = -\infty \iff \forall B < 0, \exists ? n_B \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_B \Rightarrow \frac{-5n^2-2}{4n} < B$.

We have $\frac{-5n^2-2}{4n} < B \iff \frac{5n^2+2}{4n} > -B$. It is obvious that; for all $n \in \mathbb{N}^*$ we have $5n^2+2 > 5n^2$, which leads to $\frac{5n^2+2}{4n} > \frac{5n}{4}$. Thus, for $\frac{5n^2+2}{4n} > -B$ it is sufficient to take $\frac{5n}{4} > -B \iff n > \frac{-4B}{5}$. For this, take $n_B = \lceil -4B/5 \rceil + 1$.

(6) $\lim_{n\to+\infty} \ln(\ln(n)) = +\infty \iff (\forall A > 0, \exists ? n_A \in \mathbb{N}, \forall n \in \mathbb{N}; n \geq n_A \Rightarrow \ln(\ln(n)) > A)$

For this, take $n_A = \lceil e^{e^A} \rceil + 1$.

Exercise 04:

(v_n) increasing $\iff \forall n > 0;\ v_{n+1} \geq v_n \iff v_{n+1} - v_n \geq 0$. So we have

$$\begin{aligned}
v_{n+1} - v_n &= \frac{u_1 + u_2 + \ldots + u_{n+1}}{n+1} - \frac{u_1 + u_2 + \ldots + u_n}{n} \\
&= \frac{(nu_1 + nu_2 + \ldots + nu_n) + nu_{n+1}}{n(n+1)} - \frac{(nu_1 + nu_2 + \ldots + nu_n) + u_1 + u_2 + \cdots + u_n}{n(n+1)} \\
&= \frac{-u_1 - u_2 - \ldots - u_n + nu_{n+1}}{n(n+1)} \\
&= \frac{(u_{n+1} - u_1) + (u_{n+1} - u_2) + \ldots + (u_{n+1} - u_n)}{n(n+1)}
\end{aligned}$$

Since the sequence (u_n) is increasing, for all integers k, $k = 1, 2, \ldots, n$, $u_k \leq u_{n+1}$, and thus $v_{n+1} - v_n \geq 0$. Therefore, the sequence (v_n) is increasing.