Chapter 02: Sequences of Real Numbers

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First Year Engineering

Module: Analysis 1

Semester 1

Academic Year: 2023/2024

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References

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Definitions:

- **Definitions**: A real sequence $(u_n)_{n\in\mathbb{N}}$ is defined by a function u from the set of natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$.

$$u : \mathbb{N} \to \mathbb{R} \tag{1}$$
$$n \mapsto u(n) = u_n \tag{2}$$

In this chapter we define $\mathbb{N} := \{0, 1, 2, ..\}$

  - u_n is called **the general term** of the sequence $(u_n)_{n\in\mathbb{N}}$.
  - u_0 is called **the first term** of the sequence.
  - $(u_n)_{n\in\mathbb{N}}$ is called **an arithmetic sequence** if there exists $a \in \mathbb{R}$ such that $u_{n+1} - u_n = a$. In this case, we have $u_n = u_0 + na$ for all $n \in \mathbb{N}$.
  - $(u_n)_{n\in\mathbb{N}}$ is called **a geometric sequence** if there exists $a \in \mathbb{R}$ such that $\frac{u_{n+1}}{u_n} = a$. In this case, we have $u_n = u_0 \cdot a^n$ for all $n \in \mathbb{N}$.

Monotony of a Real Sequence

Definition: Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

- $(u_n)_{n\in\mathbb{N}}$ is called **increasing** (or **strictly increasing**) if:
  $\forall n \in \mathbb{N}, u_{n+1} - u_n \geq 0$ (or $\forall n \in \mathbb{N}, u_{n+1} - u_n > 0$).
- $(u_n)_{n\in\mathbb{N}}$ is called **decreasing** (or **strictly decreasing**) if:
  $\forall n \in \mathbb{N}, u_{n+1} - u_n \leq 0$ (or $\forall n \in \mathbb{N}, u_{n+1} - u_n < 0$).
- $(u_n)_{n\in\mathbb{N}}$ is called **monotonic** if it is either **increasing** or **decreasing**.
- $(u_n)_{n\in\mathbb{N}}$ is called **strictly monotonic** if it is either **strictly increasing** or **strictly decreasing**.

Examples

1. For $u_n = n^2$, $n \in \mathbb{N}$, the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing. In fact, $u_{n+1} - u_n = (n+1)^2 - n^2 = n^2 + 1 \geq 0$ for all $n \in \mathbb{N}$.
2. For $u_n = \frac{1}{n!}$, $n \in \mathbb{N}$, the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing. In fact, $u_{n+1} - u_n = -\frac{n}{(n+1)!} \leq 0$ for all $n \in \mathbb{N}$.

Definition Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

- $(u_n)_{n\in\mathbb{N}}$ is called upper bounded if: $\exists M \in \mathbb{R}, \forall n \in \mathbb{N}, u_n \leq M$.
- $(u_n)_{n\in\mathbb{N}}$ is called lower bounded if: $\exists m \in \mathbb{R}, \forall n \in \mathbb{N}, m \leq u_n$.
- $(u_n)_{n\in\mathbb{N}}$ is called bounded if it is both upper bounded and lower bounded, or if there exists $P > 0$ such that $|u_n| \leq P$.

Examples

1. If $\forall n \in \mathbb{N}, u_n = \sin(n)$, then the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. Indeed, $|u_n| \leq 1$ for all $n \in \mathbb{N}$.
2. The sequence $(u_n)_{n\in\mathbb{N}}$; where $u_n = n^3$ is bounded below by 0 but it is not bounded above.

Subsequences

Definition: Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence and φ be a strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$. The sequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ is called **a subsequence** or **an extracted sequence** of $(u_n)_{n\in\mathbb{N}}$.

Subsequences

Example: Let $(u_n)_{n\in\mathbb{N}^*}$ be a real sequence defined by $u_n = (-1)^n \frac{1}{n}$. We can extract two subsequences $(u_{2n})_{n\in\mathbb{N}^*}$ and $(u_{2n+1})_{n\in\mathbb{N}}$ such that:

$$u_{2n} = \frac{1}{2n}, \forall n \in \mathbb{N}^*$$

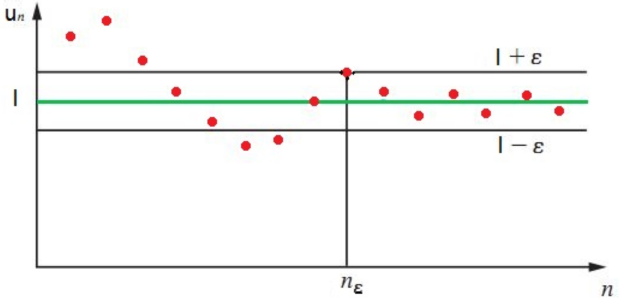
$$u_{2n+1} = -\frac{1}{2n+1}$$

Convergence of a Sequence:

Definition Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence. We say that $(u_n)_{n\in\mathbb{N}}$ is convergent if there exists a real number $l \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$, implies $|u_n - l| < \varepsilon$. We denote this as:

$$\lim_{n\to+\infty} u_n = l$$

and we say that l is the limit of $(u_n)_{n\in\mathbb{N}}$.



Example Consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = 1 - \frac{2}{5n}$.
Let's show that $(u_n)_{n\in\mathbb{N}}$ converges to 1.
$(\lim_{n\to+\infty} u_n = 1) \Leftrightarrow$
$(\forall\varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_\varepsilon \Rightarrow |u_n - 1| < \varepsilon)$

$$|u_n - 1| < \varepsilon \Leftrightarrow \frac{2}{5n} < \varepsilon \Leftrightarrow n > \frac{2}{5\varepsilon}$$

So, it suffices to take $n_\varepsilon = \left\lceil \frac{2}{5\varepsilon} \right\rceil + 1$.

Theorem If $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence, then its limit is unique.

Proof: Let's assume by contradiction that $(u_n)_{n\in\mathbb{N}}$ converges to two different limits l_1 and l_2 such that $l_1 \neq l_2$. Then we have:

$(\lim_{n\to+\infty} u_n = l_1) \Rightarrow$
$\left(\forall\varepsilon > 0, \exists n_{\varepsilon_1} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_1} \Rightarrow |u_n - l_1| < \frac{\varepsilon}{2}\right)$
$(\lim_{n\to+\infty} u_n = l_2) \Rightarrow$
$\left(\forall\varepsilon > 0, \exists n_{\varepsilon_2} \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_{\varepsilon_2} \Rightarrow |u_n - l_2| < \frac{\varepsilon}{2}\right)$

Now, let $n_{\varepsilon_0} = \max(n_{\varepsilon_1}, n_{\varepsilon_2})$, then for all $n \geq n_{\varepsilon_0}$, we have:

$$|l_2 - l_1| = |(u_n - l_1) + (l_2 - u_n)| \leq |(u_n - l_1)| + |(u_n - l_2)| < \varepsilon$$

This leads to $|l_2 - l_1| < \varepsilon$. Regardless of how small the positive number ε, this statement holds true. So, ε must be zero , which contradicts the assumption $l_1 \neq l_2$. Therefore, $l_1 = l_2$, which is absurd.

Remark: A sequence is said to be divergent if it tends towards infinity, or if it has multiple different limits.

Divergent Sequences

Definition: Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.

- $\lim_{n\to+\infty} u_n = +\infty$ if and only if $\forall A > 0, \exists n_A \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_A \Rightarrow u_n > A$.
- $\lim_{n\to+\infty} u_n = -\infty$ if and only if $\forall B < 0, \exists n_B \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_B \Rightarrow u_n < B$.

Proposition: If $(u_n)_{n\in\mathbb{N}}$ is a divergent sequence such that $\lim_{n\to+\infty} u_n = +\infty$ (resp. $\lim_{n\to+\infty} u_n = -\infty$), and $(v_n)_{n\in\mathbb{N}}$ is a sequence such that $u_n \leq v_n$ (resp. $u_n \geq v_n$) for all $n \in \mathbb{N}$, then the sequence $(v_n)_{n\in\mathbb{N}}$ is divergent and we have $\lim_{n\to+\infty} v_n = +\infty$ (resp. $\lim_{n\to+\infty} v_n = -\infty$).

Proof: Indeed, for every $A > 0$, there exists $n_A \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_A \Rightarrow u_n > A$ and $u_n \leq v_n$ for all $n \in \mathbb{N}$. Therefore, for every $A > 0$, there exists $n_A \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_A \Rightarrow v_n > A$, which implies $\lim_{n\to+\infty} v_n = +\infty$.

Proposition Every convergent sequence is bounded.
Remarks:

1. By contrapositive, an unbounded sequence is divergent.
2. The converse is not always true; a bounded sequence is not always convergent.

Example Let $u_n = (-1)^n$ for all $n \in \mathbb{N}$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded because for all $n \in \mathbb{N}$, $|(-1)^n| \leq 1$. However, $(u_n)_{n\in\mathbb{N}}$ is divergent because it has two different limits:
$\lim_{n\to+\infty} u_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

Proposition If $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence, then all its subsequences converge to the same limit.

Remark: By contrapositive, it is sufficient to find two subsequences that do not converge to the same limit in order to conclude that a sequence is divergent.

Operations on Convergent Sequences:

Theorem: Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two sequences converging respectively to the limits l_1 and l_2, and let $\lambda \in \mathbb{R}$. Then the sequences $(u_n + v_n)_{n\in\mathbb{N}}$, $(\lambda u_n)_{n\in\mathbb{N}}$, $(u_n v_n)_{n\in\mathbb{N}}$, $\left(\frac{u_n}{v_n}\right)_{n\in\mathbb{N}}$, and $(|u_n|)_{n\in\mathbb{N}}$ also converge, and we have:

1. $\lim_{n\to+\infty}(u_n + v_n) = l_1 + l_2$.
2. $\lim_{n\to+\infty}(\lambda u_n) = \lambda l_1$.
3. $\lim_{n\to+\infty}(u_n v_n) = l_1 \cdot l_2$.
4. $\lim_{n\to+\infty} \frac{u_n}{v_n} = \frac{l_1}{l_2}$ if $l_2 \neq 0$.
5. $\lim_{n\to+\infty} |u_n| = |l_1|$.

Remarks:

1. The sum of two divergent sequences can be convergent.
2. The absolute value of a divergent sequence can be convergent.

Examples:

1. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be defined as:
   $u_n = 2n$ and $v_n = -2n + e^{-n}$ for all $n \in \mathbb{N}$.
   Both $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are divergent. However, the sequence $(u_n + v_n)_{n\in\mathbb{N}}$ is convergent because $u_n + v_n = e^{-n}$ for all $n \in \mathbb{N}$.
2. Let $u_n = (-1)^n$ for all $n \in \mathbb{N}$. The sequence $(u_n)_{n\in\mathbb{N}}$ is divergent. However, we have $|u_n| = 1$ for all $n \in \mathbb{N}$, hence the sequence $(|u_n|)_{n\in\mathbb{N}}$ is convergent.

Properties:

1. If $(u_n)_{n\in\mathbb{N}}$ is a convergent sequence such that $u_n > 0$ for all $n \in \mathbb{N}$ (resp. $u_n < 0$ for all $n \in \mathbb{N}$), then $\lim_{n\to+\infty} u_n \geq 0$ (resp. $\lim_{n\to+\infty} u_n \leq 0$).
2. If $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are two convergent sequences such that $u_n < v_n$ for all $n \in \mathbb{N}$, then $\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$.

Proof:

1. Since $u_n > 0$ for all $n \in \mathbb{N}$ and $l = \lim_{n\to+\infty} u_n$, we can show that $l \geq 0$.

By assuming the opposite $l < 0$. Let $\varepsilon = \frac{|l|}{2} > 0$, then there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon \Rightarrow |u_n - l| < \frac{|l|}{2}$, $l - \frac{|l|}{2} < u_n < l + \frac{|l|}{2} < 0$, which is absurd because $u_n > 0$ for all $n \in \mathbb{N}$.

2. Since $u_n < v_n$ for all $n \in \mathbb{N}$, let $l_1 = \lim_{n\to+\infty} u_n$ and $l_2 = \lim_{n\to+\infty} v_n$. Suppose by contradiction that $l_2 < l_1$, and let $\varepsilon = \frac{l_1 - l_2}{2} > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon \Rightarrow |u_n - l_1| < \frac{l_1 - l_2}{2}$, which implies
$\frac{l_1+l_2}{2} < u_n < \frac{3l_1 - l_2}{2} \quad (1)$.
Also, there exists $n'_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,
$n \geq n'_\varepsilon \Rightarrow |v_n - l_2| < \frac{l_1 - l_2}{2}$, leading to
$\frac{3l_2 - l_1}{2} < v_n < \frac{l_1 + l_2}{2} \quad (2)$.
Let $n''_\varepsilon = \max(n_\varepsilon, n'_\varepsilon)$. Combining (1) and (2), we have $\exists n''_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n''_\varepsilon \Rightarrow v_n < \frac{l_1+l_2}{2} < u_n$.
Therefore, $v_n < u_n$, which is absurd because $u_n < v_n$ for all $n \in \mathbb{N}$.
Alternatively, we can view this property as a direct consequence of the first one, where we simply set $w_n = v_n - u_n$. Since $w_n > 0$ for all $n \in \mathbb{N}$, we have $\lim_{n\to+\infty} w_n \geq 0$, implying $\lim_{n\to+\infty}(v_n - u_n) \geq 0$, which further leads to $\lim_{n\to+\infty} v_n \geq \lim_{n\to+\infty} u_n$.

Theorem: Any increasing (resp. decreasing) and bounded above (resp. bounded below) sequence converges to its supremum (resp. infimum).

Proof: Let $(u_n)_{n\in\mathbb{N}}$ be an increasing and bounded above sequence. Then, for all $n \in \mathbb{N}$, $u_n \leq u_{n+1}$, and there exists $M \in \mathbb{R}$ such that $u_n \leq M$. Let $E = \{u_n, n \in \mathbb{N}\}$ and $u = \sup(E)$. According to the characterization of the supremum, we have, for every $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that $u - \varepsilon < u_p$.

Since (u_n) is increasing, for all $n \in \mathbb{N}$ such that $n \geq p$, we have $u_p \leq u_n$.

Now, since $u_n \leq u$, we get $u - \varepsilon < u_p \leq u_n \leq u < u + \varepsilon$. Hence, for every $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ such that $n \geq p$, we have $|u_n - u| < \varepsilon$. Therefore, $\lim_{n\to+\infty} u_n = \sup(E)$.

Theorem: Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$, and $(w_n)_{n\in\mathbb{N}}$ be three real sequences such that for all $n \geq n_0$, $u_n \leq v_n \leq w_n$, and $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} w_n = l$, then $\lim_{n\to+\infty} v_n = l$.
Proof: Let $\varepsilon > 0$. There exists $n_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ such that $n \geq n_1$, we have $|u_n - l| < \varepsilon$ which implies $l - \varepsilon < u_n < l + \varepsilon$. Similarly, there exists $n_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ such that $n \geq n_2$, we have $|w_n - l| < \varepsilon$ which implies $l - \varepsilon < w_n < l + \varepsilon$. Let $n_3 = \max(n_0, n_1, n_2)$. Then, for all $n \in \mathbb{N}$ such that $n \geq n_3$, we have $l - \varepsilon < u_n \leq v_n \leq w_n < l + \varepsilon$, which leads to $l - \varepsilon < v_n < l + \varepsilon$ or $|v_n - l| < \varepsilon$. Therefore, for every $\varepsilon > 0$, there exists $n_3 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ such that $n \geq n_3$, we have $|v_n - l| < \varepsilon$, which concludes that $\lim_{n\to+\infty} v_n = l$.

Theorem: Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences such that $\lim_{n\to+\infty} u_n = 0$ and $(v_n)_{n\in\mathbb{N}}$ is bounded. Then $\lim_{n\to+\infty} u_n \cdot v_n = 0$.

Proof: Since $(v_n)_{n\in\mathbb{N}}$ is bounded, there exists $M > 0$ such that $|v_n| \leq M$ for all $n \in \mathbb{N}$. Also, $\lim_{n\to+\infty} u_n = 0$ implies that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ such that $n \geq n_\varepsilon$, we have $|u_n| < \frac{\varepsilon}{M}$.

This leads to $|u_n \cdot v_n| = |u_n| \cdot |v_n| < \frac{\varepsilon}{M} \cdot M = \varepsilon$. Thus, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ such that $n \geq n_\varepsilon$, we have $|u_n \cdot v_n| < \varepsilon$, which means $\lim_{n\to+\infty} u_n \cdot v_n = 0$.

Theorem (Bolzano-Weierstrass): Every bounded real sequence $(u_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

Adjacent Sequences:

Definition: Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be two real sequences, such that $(u_n)_{n\in\mathbb{N}}$ is increasing and $(v_n)_{n\in\mathbb{N}}$ is decreasing. The sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are called **adjacent** if $\lim_{n\to+\infty}(u_n - v_n) = 0$.

Theorem: Two adjacent real sequences converge to the same limit.
Example: The sequences $(u_n)_{n\in\mathbb{N}^*}$ and $(v_n)_{n\in\mathbb{N}^*}$ defined by $u_n = \sum_{k=1}^{n} \frac{1}{k!}$ and $v_n = u_n + \frac{1}{n!}$ respectively, converge to the same limit since they are adjacent. Indeed, $(u_n)_{n\in\mathbb{N}^*}$ is increasing, $(v_n)_{n\in\mathbb{N}^*}$ is decreasing, and we have
$\lim_{n\to+\infty}(v_n - u_n) = \lim_{n\to+\infty} \frac{1}{n!} = 0$.

Cauchy's Convergence Criterion

Theorem: Let $(u_n)_{n\in\mathbb{N}}$ be a convergent sequence. Then, $(u_n)_{n\in\mathbb{N}}$ possesses the following property known as the Cauchy criterion. For any $\varepsilon > 0$, there exists an integer N such that for every pair of integers p and q greater than N, we have $|u_p - u_q| < \varepsilon$.

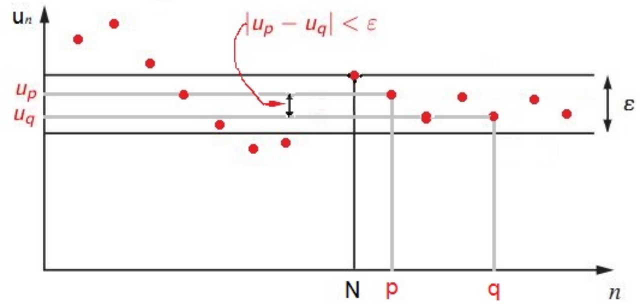


Figure:

proof:
Let l be the limit of the sequence. We have

$$|u_p - u_q| = |u_p - l + l - u_q| \leq |u_p - l| + |l - u_q|$$

The sequence $(u_n)_{n\in\mathbb{N}}$ converges to l. Therefore, by definition, for any $\varepsilon > 0$, we can associate an integer N such that for all $p > N$, we have $|u_p - l| < \frac{\varepsilon}{2}$, and for all integer $q > N$, we have $|u_q - l| < \frac{\varepsilon}{2}$. For any pair of integers p and q greater than N,

$$|u_p - u_q| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

This brings us to the following definition:
Definition: We say that a sequence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if it possesses the following property, known as the Cauchy criterion: For any $\varepsilon > 0$, there exists a natural number N such that for any pair of integers p and q greater than N, we have

$$|u_p - u_q| < \varepsilon$$

or, in short,

$$\forall\varepsilon > 0, \exists N, \forall p, \forall q, \quad (p, q > N \Rightarrow |u_p - u_q| < \varepsilon)$$

Example: Show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence where $u_n = \frac{1}{n}$.
We have $|u_p - u_q| = |\frac{1}{p} - \frac{1}{q}| \leq |\frac{1}{p}| + |-\frac{1}{q}|$. Let us take

$$\begin{cases} q > N \\ p > N \end{cases} \implies \begin{cases} \frac{1}{q} < \frac{1}{N} \\ \frac{1}{p} < \frac{1}{N} \end{cases}$$

Thus, $|u_p - u_q| \leq \frac{1}{q} + \frac{1}{p} < \frac{1}{N} + \frac{1}{N}$.
So that $|u_p - u_q| < \varepsilon$, it suffices that $\frac{2}{N} < \varepsilon$. And so it suffices to take:

$$N = \left[\frac{2}{\varepsilon}\right] + 1$$

Recursive Sequences

Definition
Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function. We call a recursive sequence a sequence (u_n) for $n \in \mathbb{N}$ defined by $u_0 \in D$ and the relation

$$\forall n \in \mathbb{N} : u_{n+1} = f(u_n).$$

In the study of recursive sequences, we always assume that $f(D) \subseteq D$.
Example: Let $u_n = 2u_{n-1} + 1$ and $u_0 = 1$ we can compute the next terms:

$$u_1 = 2 \cdot 1 + 1 = 3$$

$$u_2 = 2 \cdot 3 + 1 = 7$$

Remarks:

- If the function f is increasing, then studying the monotony of $(u_n)_{n\in\mathbb{N}}$ is given by examining the sign of the difference $f(u_0) - u_0$.
  - If $f(u_0) - u_0 > 0$, then the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing.
  - If $f(u_0) - u_0 < 0$, then the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing.
- If the function f is monotonic and continuous on D, and the sequence $(u_n)_{n\in\mathbb{N}}$ converges to a limit $l \in D$, then its limit satisfies the equation $f(l) = l$.

Computation of Limits in 'Python'

In Python, you can; for example , use the ‘sympy‘ library to perform the limit as $n \to \infty$ of the sequence defined by $u_n = f(n)$ is determined by the commands:

```
import sympy as sp
n= sp.symbols('n');
result_1= sp.limit((3*n-1)/(4*n+5), n,sp.oo)
print(result_1)
```

This code uses ‘sympy‘ to define a symbolic variable ‘n‘ and then uses ‘sp.limit‘ to calculate the limit of the sequence (u_n) where: $u_n = \frac{3n-1}{4n+5}$. The function ‘sp.oo‘ represents infinity in sympy. Using 'print(...)', the result will be:

```
3/4
```

If one takes $u_n = n$, then he can write the code:

```
import sympy as sp
n= sp.symbols('n')
result_2= sp.limit(n, n,sp.oo)
print(result_2)
```

and find the result (infinity):

```
oo
```

Thanks