Solutions of tutorial exercises set 1:

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This document is supplemented for the first chapter lecture notes (Analyses 1).

Exercise 01:

A)

1. For all real numbers x and y, we have:
$$2|x| = |(x+y)+(x-y)| \implies 2|x| \leq |x+y|+|x-y|$$
$$2|y| = |(x+y)+(y-x)| \implies 2|y| \leq |x+y|+|x-y|$$
Therefore,
$$|x|+|y| \leq |x+y|+|x-y|, \forall x,y \in \mathbb{R}.$$
2. $\forall x,y \geq 0$, we have $x+y \leq x+2\sqrt{xy}+y$; because $2\sqrt{xy} \geq 0$, which leads to $x+y \leq (\sqrt{x}+\sqrt{y})^2$ we find$\sqrt{x+y} \leq \sqrt{x}+\sqrt{y}$.
3. For all $x,y \geq 0$ such that $x=(x-y)+y$ and $(x-y)+y \leq |x-y|+y$, so we have
$$\sqrt{x} \leq \sqrt{|x-y|+y}.$$
Using the result in question 2, we find $\sqrt{x} \leq \sqrt{|x-y|}+\sqrt{y}$, this implies
$$\sqrt{x}-\sqrt{y} \leq \sqrt{|x-y|}. \tag{1}$$
Similarly, we have $\sqrt{y} \leq \sqrt{|y-x|+x}$, and using the question 2, we get $\sqrt{y} \leq \sqrt{|x-y|}+\sqrt{x}$, which implies:
$$\sqrt{x}-\sqrt{y} \geq -\sqrt{|x-y|} \tag{2}$$
Combining equations (1) and (2), we get $-\sqrt{|x-y|} \leq \sqrt{x}-\sqrt{y} \leq |x-y|$, which implies $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|}$.

B)

1. For any $x \in \mathbb{R}$, we have
$$[x] \leq x \leq [x]+1,$$
which implies
$$[x]+m \leq x+m \leq [x]+m+1,$$
for all $m \in \mathbb{Z}$. On the other hand,
$$[x+m] \leq x+m \leq [x+m]+1,$$
since $[x+m]$ is the largest integer less then $x+m$, we have
$$[x]+m \leq [x+m]$$
Similarly, $[x+m]+1$ is the smallest integer greater than or equal to $x+m$, so
$$[x+m]+1 \leq [x]+m+1$$
Combining these, we get
$$[x+m] \leq [x]+m$$
From $[x]+m \leq [x+m]$ and $[x+m] \leq [x]+m$, we conclude $[x+m]=[x]+m$.

2. If $x \leq y$, then $[x] \leq x \leq y < [y]+1$, so $[x] \leq y < [y]+1$. As $[y]$ is the greatest integer less than or equal to y and $[x]$ is an integer, we have $[x] \leq [y]$.

3. $[x] \leq x < [x]+1$ and $[y] \leq y < [y]+1$ imply $[x]+[y] \leq x+y < [x]+[y]+2$. Since $[x+y]$ is the greatest integer less than or equal to $x+y$, we get

$$[x]+[y] \leq [x+y]. \tag{3}$$

Also, $[x+y]+1$ is the smallest integer greater than $x+y$, so $[x+y]+1 \leq [x]+[y]+2$, leading to

$$[x+y] \leq [x]+[y]+1. \tag{4}$$

From (3) and (4), we find

$$[x]+[y] \leq [x+y] \leq [x]+[y]+1.$$

Exercise 02:

A)

1. Let $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$. We assume by contradiction that $z = x+y \in \mathbb{Q}$, which implies $y = z - x \in \mathbb{Q}$, leading to a contradiction.

2. Look at the solution for exercise 7 from the solutions of tutorial exercises set 0.

B) We have:

$$x^2 = 2a + 2|a-2| = \begin{cases} 4a-4, & \text{if } a \geq 2 \\ 4, & \text{if } 1 \leq a \leq 2 \end{cases}$$

Exercise 03:

1.

A	Maj(A)	Min(A)	$\sup A$	$\inf A$	max$A$	min$A$
$[-\alpha,\alpha]$	$[\alpha,+\infty[$	$]-\infty,-\alpha]$	α	$-\alpha$	α	$-\alpha$
$[-\alpha,\alpha[$	$[\alpha,+\infty[$	$]-\infty,-\alpha]$	α	$-\alpha$	$\nexists$	$-\alpha$
$]-\alpha,\alpha]$	$[\alpha,+\infty[$	$]-\infty,-\alpha]$	α	$-\alpha$	α	$\nexists$
$]-\alpha,\alpha[$	$[\alpha,+\infty[$	$]-\infty,-\alpha]$	α	$-\alpha$	$\nexists$	$\nexists$

2. $A = [-\sqrt{2}, \sqrt{2}[$, (4th case in the above table).

3. $A = \{\frac{n-1}{n}, \text{where } n \in \mathbb{N}^*\}$. For all $n \in \mathbb{N}^* : n \geq 1 \Leftrightarrow n-1 \geq 0 \Rightarrow \frac{n-1}{n} \geq 0$ and $0 \in A$, hence $\min A = \inf A = 0$.

$$\sup A = 1 \Leftrightarrow \begin{cases} \text{(a)} \ \forall n \in \mathbb{N}^*, \frac{n-1}{n} \leq 1. \\ \text{(b)} \ \forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N}^* : 1-\varepsilon < \frac{n_\varepsilon - 1}{n_\varepsilon}. \end{cases}$$

Let us discuss these two conditions:

(a) $\forall n \in \mathbb{N}^*, n-1 \leq n \Leftrightarrow \frac{n-1}{n} \leq 1$.

(b) Let $\varepsilon > 0$, $1-\varepsilon < \frac{n-1}{n} \Leftrightarrow 1-\varepsilon < 1-\frac{1}{n} \Leftrightarrow \varepsilon > \frac{1}{n} \Leftrightarrow n > \frac{1}{\varepsilon}$

Then the condition related to n and ε, suggesting that n_ε can be taken as $\left[\frac{1}{\varepsilon}\right]+1$.

Exercise 04:

$$B = \{|x-y|; (x,y) \in A^2\}.$$

1. If A is a bounded subset, then $\sup A$ and $\inf A$ exist. Let $\sup A = M$ and $\inf A = m$. For all (x,y) in A^2: let us take, $m \leq x \leq M$ and $m \leq y \leq M$, which leads to $-M \leq -y \leq -m \Rightarrow -(M-m) \leq x-y \leq M-m$

$$\Leftrightarrow |x-y| \leq M-m.$$

Therefore, $M-m$ is an upper bound for B.
2. We have

$$\text{If } \sup A = M, \text{ then for all } \varepsilon > 0, \text{ there exists } x \in A \text{ such that } M - \frac{\varepsilon}{2} < x \tag{5}$$

and

$$\text{If } \inf A = m, \text{ then for all } \varepsilon > 0, \text{ there exists } y \in A \text{ such that } y < m + \frac{\varepsilon}{2} \tag{6}$$

Combining (5) and (6), we get:

$$\forall \varepsilon > 0, \exists (x,y) \in A^2, (M-m) - \varepsilon < x - y$$

Since $x - y \leq |x-y|$, we have:

$$\forall \varepsilon > 0, \exists (x,y) \in A^2, (M-m) - \varepsilon < |x - y|$$

Consequently, $\sup B = M - m = \sup A - \inf A$.

Exercise 05:

1. (a) Let us show that: $\sup(A \cup B) \stackrel{?}{=} \max(\sup A, \sup B)$. We have on one hand:

$$\begin{cases} A \subset (A \cup B) \\ \text{and} \\ B \subset (A \cup B) \end{cases}$$

This implies:

$$\begin{cases} \sup A \leq \sup(A \cup B) \\ \text{and} \\ \sup B \leq \sup(A \cup B) \end{cases}$$

Therefore,

$$\max(\sup A, \sup B) \leq \sup(A \cup B) \tag{7}$$

On the other hand, if $x \in A \cup B$, then:

$$\begin{cases} x \in A \\ \text{or} \\ x \in B \end{cases}$$

This leads to:

$$\begin{cases} x \leq \sup A \\ \text{or} \\ x \leq \sup B \end{cases}$$

So, $x \leq \max(\sup A, \sup B)$, implying that $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$. Since $\sup(A \cup B)$ is the smallest upper bound for $A \cup B$, we have

$$\sup(A \cup B) \leq \max(\sup A, \sup B) \tag{8}$$

Combining (7) and (8), we establish the equality.

(b) Let us show that: $\inf(A \cup B) \stackrel{?}{=} \min(\inf A, \inf B)$. On one hand:

$$\begin{cases} A \subset (A \cup B) \\ \text{and} \\ B \subset (A \cup B) \end{cases}$$

This implies:

$$\begin{cases} \inf A \geq \inf(A \cup B) \\ \text{and} \\ \inf B \geq \inf(A \cup B) \end{cases}$$

Therefore,

$$\min(\inf A, \inf B) \geq \inf(A \cup B) \tag{9}$$

On the other hand, if $x \in A \cup B$, then:

$$\begin{cases} x \in A \\ \text{or} \\ x \in B \end{cases}$$

This leads to:

$$\begin{cases} x \geq \inf A \\ \text{or} \\ x \geq \inf B \end{cases}$$

So, $x \geq \min(\inf A, \inf B)$, implying that $\min(\inf A, \inf B)$ is a lower bound for $A \cup B$. Since $\inf(A \cup B)$ is the largest lower bound for $A \cup B$, we have

$$\inf(A \cup B) \geq \min(\inf A, \inf B) \tag{10}$$

Combining (9) and (10), we establish the equality.

2. If $A \cap B \neq \emptyset$, then, let us prove that:

(a)

$$\sup(A \cap B) \stackrel{?}{\leq} \min(\sup A, \sup B)$$

$$\begin{cases} (A \cap B) \subset A \\ \text{and} \\ (A \cap B) \subset B \end{cases}$$

This implies:

$$\begin{cases} \sup(A \cap B) \leq \sup A \\ \text{and} \\ \sup(A \cap B) \leq \sup B \end{cases}$$

Hence, $\sup(A \cap B) \leq \min(\sup A, \sup B)$.

(b)

$$\inf(A \cap B) \stackrel{?}{\geq} \max(\inf A, \inf B)$$

Let us take

$$\begin{cases} (A \cap B) \subset A \\ \text{and} \\ (A \cap B) \subset B \end{cases}$$

This implies:

$$\begin{cases} \inf(A \cap B) \geq \inf A \\ \text{and} \\ \inf(A \cap B) \geq \inf B \end{cases}$$

Thus, $\inf(A \cap B) \geq \max(\inf A, \inf B)$.

3. (a) Let us show that:

$$\sup(A+B) \stackrel{?}{=} \sup A + \sup B$$

Given:

$$\sup A = M_A \implies \begin{cases} \forall x \in A : x \leq M_A ...(*1) \\ \forall \varepsilon > 0, \exists x \in A : M_A - \frac{\varepsilon}{2} < x...(*2) \end{cases}$$

$$\sup B = M_B \implies \begin{cases} \forall y \in B : y \leq M_B ...(*3) \\ \forall \varepsilon > 0, \exists y \in B : M_B - \frac{\varepsilon}{2} < y...(*4) \end{cases}$$

Then:

$$\begin{aligned} (*1) + (*3) &\implies \forall z \in A + B : z \leq M_A + M_B \\ (*2) + (*4) &\implies \forall \varepsilon > 0, \exists z \in A + B : (M_A + M_B) - \varepsilon < z \end{aligned}$$

Therefore, $\sup(A+B) = \sup A + \sup B$.

(b) Now, let us show that:

$$\inf(A+B) \stackrel{?}{=} \inf A + \inf B$$

Given:

$$\inf A = m_A \implies \begin{cases} \forall x \in A : m_A \leq x...(**1) \\ \forall \varepsilon > 0, \exists x \in A : x < m_A + \frac{\varepsilon}{2}...(**2) \end{cases}$$

$$\inf B = m_B \implies \begin{cases} \forall y \in B : m_B \leq y...(**3) \\ \forall \varepsilon > 0, \exists y \in B : y < m_B + \frac{\varepsilon}{2}...(**4) \end{cases}$$

Then:

$$\begin{aligned} (**1) + (**3) &\implies \forall z \in A + B : m_A + m_B \leq z \\ (**2) + (**4) &\implies \forall \varepsilon > 0, \exists z \in A + B : z < (m_A + m_B) + \varepsilon \end{aligned}$$

Therefore, $\inf(A+B) = \inf A + \inf B$.

4. (a) Let us show that:

$$\sup(-A) \stackrel{?}{=} -\inf A$$

- $\forall x \in A : x \geq \inf A \implies -x \leq -\inf A$. Hence, $-\inf A$ is an upper bound for $-A$. Since $\sup(-A)$ is the smallest upper bound for $-A$, we have

$$\sup(-A) \leq -\inf A \tag{11}$$

- $\forall (-x) \in (-A) : -x \leq \sup(-A) \implies x \geq -\sup(-A)$. Hence, $-\sup(-A)$ is a lower bound for A. Since $\inf A$ is the largest lower bound for A, we have

$$\inf A \geq -\sup(-A) \tag{12}$$

From (11) and (12), we establish the equality.

(b) Let us show that:

$$\inf(-A) \stackrel{?}{=} -\sup A$$

- $\forall x \in A : x \leq \sup A \implies -x \geq -\sup A$. Hence, $-\sup A$ is a lower bound for $-A$. Since $\inf(-A)$is the largest lower bound for $-A$, we have

$$\inf(-A) \geq -\sup A \tag{13}$$

- $\forall(-x) \in (-A) : -x \geq \inf(-A) \iff x \leq -\inf(-A)$. Hence, $-\inf A$ is an upper bound for A. Since $\sup A$ is the smallest upper bound for A, we have $\sup A \leq -\inf(-A)$, which leads

$$-\sup A \geq \inf(-A) \tag{14}$$

From (13) and (14), we establish the equality.

Exercise 06:

1. $A = \left\{\frac{3n+1}{2n+1}, n \in \mathbb{N}\right\}$

   - Let us show that: $\inf A \stackrel{?}{=} 1$. We have

   $$\forall n \in \mathbb{N} : 3n \geq 2n \iff 3n+1 \geq 2n+1 \iff \frac{3n+1}{2n+1} \geq 1$$

   So, 1 is a lower bound for A. Note: $1 \in A$ for $n = 0$. Thus, $\min A = \inf A = 1$.

   - $\sup A \stackrel{?}{=} \frac{3}{2}$. We have

   $$\forall n \in \mathbb{N} : 2 < 3 \implies 6n+2 < 6n+3 \iff \frac{3n+1}{2n+1} < \frac{3}{2}$$

   Therefore, $\frac{3}{2}$ is an upper bound for A; but $\frac{3}{2} \notin A$. The verification of the supremum characterization leads to: For any $\varepsilon > 0$, there exists (?) $n_\varepsilon \in \mathbb{N}$ such that $\frac{3}{2} - \varepsilon < \frac{3n_\varepsilon+1}{2n_\varepsilon+1}$. We have: $\frac{3}{2} - \varepsilon < \frac{3n_\varepsilon+1}{2n_\varepsilon+1}$, which implies $\left(\frac{3}{2} - \varepsilon\right)(2n_\varepsilon + 1) < (3n_\varepsilon + 1) \implies \left(3 - 2\varepsilon\right)(2n_\varepsilon + 1) < (6n_\varepsilon + 2)$, then $(6n_\varepsilon + 3 - 4\varepsilon n_\varepsilon - 2\varepsilon) < (6n_\varepsilon + 2) \implies 1 < 2\varepsilon(2n_\varepsilon + 1)$. Hence, $\frac{1-2\varepsilon}{\varepsilon} < n_{4\varepsilon}$.
   Choose $n_\varepsilon = \left[\frac{1-2\varepsilon}{4\varepsilon}\right] + 1$. Thus, $\sup A = \frac{3}{2}$, but $\frac{3}{2} \notin A$, so $\max A$ does not exist.

2. $B = \left\{\frac{1}{n} + \frac{1}{n^2}, n \in \mathbb{N}^*\right\}$

   - $\sup B \stackrel{?}{=} 2$. We have $\forall n \in \mathbb{N}^*$:

   $$\begin{cases} n \geq 1 \\ n^2 \geq 1 \end{cases} \implies \begin{cases} 1 \geq \frac{1}{n} \\ 1 \geq \frac{1}{n^2} \end{cases} \implies 2 \geq \frac{1}{n} + \frac{1}{n^2}$$

   Hence, 2 is an upper bound for B. Note: $2 \in B$ for $n = 1$. Thus, $\max B = \sup B = 2$.

   - $\inf B \stackrel{?}{=} 0$. We have

   $$\forall n \in \mathbb{N}^* : \frac{1}{n} + \frac{1}{n^2} > 0$$

   So, 0 is a lower bound for B. For any $\varepsilon > 0$, there exists (?) $n_\varepsilon \in \mathbb{N}^*$ such that $\frac{1}{n_\varepsilon} + \frac{1}{n_\varepsilon^2} < \varepsilon$.

   Let us take $\varepsilon > 0$, then we have $\forall n \in \mathbb{N}^* : n + 1 \leq 2n \iff \frac{n+1}{n^2} \leq \frac{2n}{n^2}$, which leads to $\frac{1}{n} + \frac{1}{n^2} \leq \frac{2}{n}$. So for $\frac{1}{n} + \frac{1}{n^2} \leq \varepsilon$, it is sufficient to take: $\frac{2}{n} < \varepsilon \iff \frac{2}{\varepsilon} < n$. Choose $n_\varepsilon = \left[\frac{2}{\varepsilon}\right] + 1$. Thus, $\inf B = 0$, but $0 \notin B$, so $\min B$ does not exist.

3. $C = \{e^{-n}, n \in \mathbb{N}\}$

   - $\sup C \stackrel{?}{=} 1$ (as e^{-n} approaches 0 for increasing n).
   For all $n \in \mathbb{N}$: $0 \leq n \iff -n \leq 0 \iff e^{-n} \leq 1$, then 1 is an upper bound for C.
   Note that $1 \in C$ for $n = 0$, so $\max C = \sup C = 1$.

   - $\inf C \stackrel{?}{=} 0$ (trivial since e^{-n} is always positive)
   For all $n \in \mathbb{N}$: $e^{-n} > 0$, so 0 is a lower bound for C.
   For any $\varepsilon > 0$, there exists (?) $n_\varepsilon \in \mathbb{N}$ such that $e^{-n_\varepsilon} < \varepsilon$.
   Let $\varepsilon > 0$, then $e^{-n} < \varepsilon \iff -n < \ln(\varepsilon) \iff -\ln(\varepsilon) < n$. It suffices to take $n_\varepsilon = \left[-\ln(\varepsilon)\right] + 1$.
   Therefore, $\inf C = 0$, but $0 \notin C$ so $\min C$ does not exist.

4. $D = \{\frac{1}{n^2} - 2, n \in \mathbb{N}^*\}$

- $\sup D \stackrel{?}{=} -1$

  For all $n \in \mathbb{N}^*$: $1 \leq n \iff 1 \leq n^2 \iff \frac{1}{n^2} \leq 1 \iff \frac{1}{n^2} - 2 \leq -1$, so -1 is an upper bound for D.
  Note that $-1 \in D$ for $n = 1$, so $\max D = \sup D = -1$.

- $\inf D \stackrel{?}{=} -2$

  For all $n \in \mathbb{N}^*$: $0 < \frac{1}{n^2} \iff -2 < \frac{1}{n^2} - 2$, so -2 is a lower bound for D.
  For any $\varepsilon > 0$, there exists (?) $n_\varepsilon \in \mathbb{N}^*$ such that $\frac{1}{n^2} - 2 < \varepsilon - 2$.
  Let $\varepsilon > 0$, then $\frac{1}{n^2} - 2 < \varepsilon - 2 \iff \frac{1}{n^2} < \varepsilon \iff \frac{1}{\varepsilon} < n^2 \iff \frac{1}{\sqrt{\varepsilon}} < n$; since $n \in \mathbb{N}$, it suffices to take $n_\varepsilon = \left[\frac{1}{\sqrt{\varepsilon}}\right] + 1$.
  Therefore, $\inf D = -2$, but $-2 \notin D$ so $\min D$ does not exist.