Solutions to tutorial exercises set(0) : (Analysis 1)

Exercise 01:

A. All these numbers are real, so they are d:

(i) -2 is a negative natural number, i.e. an integer (a, b, c, d)

(ii) $\frac{1}{3}$ is a ratio of integers so it is a rational (c, d)

(iii) 0 is an integer (a, c, d).

(iv) 7 is a natural number and an integer. It is in fact also a prime number - That is, only divisible by itself or 1. It is also an odd number (can not be exactly divided by 2)

(a, c, d, g)

(V) $\frac{21}{5}$ is a rational number (actually an improper fraction) (c, d).

$|\text{numerator}| > |\text{denominator}|$
(21) (5)

(vi) $-\frac{3}{4}$ is a rational number (a proper fraction) (b, c, d)

$|\text{numerator}| < |\text{denominator}|$
$|-3| < |4|$

(vii) 0,73 is a actually a decimal representation of a rational number $\left(0.73 = \frac{73}{100}\right)$ sometimes called a decimal fraction, or simply a decimal; (c, d, f).

(viii) 11 is a natural number and an integer (like 7 it is also prime, and is also odd as any prime greater than 2 must be,

(a, c, d, g)

(ix) 8 is another natural number and an integer (but it is not prime) since it can be written as $2\times2\times2 = 2^3$. It is also an even number

(a, c, d).

(x) The square root of 2 is not a rational number. (d, e)

(xi) -0.49 is a decimal representation of the negative rational number $\left(-\frac{49}{100}\right)$ (b, c, d, f)

(xii) π, the ratio of the circumference of a circle to its diameter, is not a rational number (it is an irrational number). That is, it can not be written as a fraction. $\frac{22}{7}$ for example, is just an approximation to π; (d, e).

B. (i) $0\times1=0$, i.e. zero - which is also finite

(ii) $0+1=1$, finite, non-zero.

(iii) $\frac{1}{0}$ does not exist - it is not infinite, negative, zero, finite or non-zero (b).

(iv) $2-0=2$, finite and non-zero (e, f).

(v) $0^2=0\times0=0$, zero and finite (d, e).

(vi) $0-1=-1$, negative, finite, non-zero (c, e, f).

(vii) $\frac{0}{0}$ does not exist (you can't "cancel" the zeros!). It is not infinite, negative, zero, finite or non-zero - it just does not exist (b).

(viii) $3\times0+\frac{3}{0}$ does not exist because $\frac{3}{0}$ does not exist (b).

(ix) $\frac{0^3}{0}$ again, does not exist (b)

(x) $\frac{2}{2}=1$ - no problem here, finite and non-zero (e, f)

Exercise 02:

1) $x>0$

2) $1<x<0$

3) $-1<x<3$

4) $-2\leq x<2$

5) If the absolute value of x is less than 2 then this means that if x is positive then $0\leq x<2$, but if x is negative then we must have $-2<x\leq0$. So, combining these we must have $-2<x<2$. This can also be expressed in terms of modulus $|x|<2$.

Exercise 03:

(i) $$\frac{2}{x+1}-\frac{3}{x-2}=\frac{2(x-2)-3(x+1)}{(x+1)(x-2)}$$

$$=\frac{2x-4-3x-3}{x^2-2x+x-2}$$

$$=\frac{-x-7}{x^2-x-2}$$

$$=-\frac{x+7}{x^2-x-2}$$

(ii) $$\frac{1}{x-1}+\frac{1}{x+1}-\frac{1}{x+2}=\frac{(x+1)+(x-1)}{(x-1)(x+1)}-\frac{1}{(x+2)}$$

$$=\frac{x+1+x-1}{(x^2-1)}-\frac{1}{(x+2)}$$

$$=\frac{2x(x+2)-(x^2-1)}{(x^2-1)(x+2)}$$

$$=\frac{2x^2+4x-x^2+1}{x^3+2x^2-x-2}$$

$$=\frac{x^2+4x+1}{x^3+2x^2-x-2}$$

Exercise 04:

odd function : $f(-x)=-f(x)$

even function : $f(-x)=f(x)$.

(i) $3x^3-x=f(x) \Rightarrow f(-x)=3(-x)^3-(-x)=-(3x^3-x)=-f(x)$

$3x^3-x$ is an odd function

(ii) $$\frac{x^2}{1+x^2}=f(x) \Rightarrow f(-x)=\frac{(-x)^2}{1+(-x)^2}=f(x)$$

$\frac{x^2}{1+x^2}$ is an even function

(iii) $$\frac{2x}{x^2-1}=f(x) \Rightarrow f(-x)=\frac{2(-x)}{(-x)^2-1}=-\frac{2x}{x^2-1}=-f(x)$$

$\frac{2x}{1+x^2}$ is an odd function.

(iv) $\frac{x^2}{x-1} = f(x) \Rightarrow f(-x) = \frac{(-x)^2}{(-x)-1} = \frac{x^2}{-x-1}$

$\frac{x^2}{x-1}$ neither an even function nor an odd function.

Exercise 05:

(a) Let us suppose $\frac{a}{b}$ as a number (if it exists) such that $bx = a$, then $\frac{0}{0} = x$; where $0x = 0$ but this is true for all numbers.
$\Rightarrow$ it is undefined.

(b) Similar to (a), if we define $\frac{1}{0}$ as a number (if it exists) such that $0 \cdot x = 1$, we reach the conclusion that there is no satisfied this condition.

Exercise 06:

An odd integer: $n = 2k+1$ where $k \in \mathbb{Z}$

we have: $(2k+1)^2 = 4k^2 + 4k + 1$. Notice that, we can write:

$4k^2 + 4k = 2(2k^2 + 2k) = 2p$ where $p = 2k^2 + 2k$

So: $(2k+1)^2 = 2p+1$, which is an odd number.

Exercise 07:

We assume by contradiction that $\sqrt{2} \in \mathbb{Q}$ then:

$\exists\, p, q \in \mathbb{Z}$, $GCD(p,q) = 1$ such that $\sqrt{2} = \frac{p}{q} \Leftrightarrow 2q^2 = 2$ divides p

greatest common divisor

$\Rightarrow$ 2 divide p because 2 is a prime number, so $p = 2k$ where $k \in \mathbb{Z}$, and $4k^2 = 2q^2 \Rightarrow 2k^2 = q^2 \Rightarrow$ 2 divides $q^2 \Rightarrow$ 2 divides q, which is a contradiction since p and q are relatively prime.

Exercise 08:

Since $\frac{p}{q}$ is a root, on substituting in the given equation, we find:

$a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + a_2\left(\frac{p}{q}\right)^{n-2} + \dots + a_{n-1}\left(\frac{p}{q}\right) + a_n = 0$

multiplying by q^n: $\boxed{a_0 p^n + a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \dots + a_{n-1} p q^{n-1} + a_n q^n = 0}$ ---(*)

dividing by p: $a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1} = -\frac{a_n q^n}{p}$

Notice that, $a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1}$ is an integer

$\Rightarrow -\frac{a_n q^n}{p}$ is also integer. Then since p and q are relatively prime
$\Rightarrow p$ does not divide q^n exactly and so must divide a_n

In the other hand, by transposing the first term of (*):

$$a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \dots + a_{n-1} p q^{n-1} + a_n q^n = -a_0 p^n$$

and dividing by q:

$$a_1 p^{n-1} + a_2 p^{n-2} q + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1} = -\frac{a_0 p^n}{q}$$

we find that: $a_1 p^{n-1} + a_2 p^{n-2} q + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1} \in \mathbb{Z}$

so $-\frac{a_0 p^n}{q} \in \mathbb{Z}$

$\Rightarrow q$ must divide a_0 (since p and q are relatively prime so q does not divide p^n exactly and so must divide a_0)

Exercise 09:

Let us take: $x = \sqrt{2} + \sqrt{3} \Rightarrow x^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$

$\Leftrightarrow x^2 - 5 = 2\sqrt{6}$ and squaring $(x^2-5)^2 = x^4 - 10x^2 + 25 = 4 \times 6 = 24$

$\Rightarrow x^4 - 10x^2 + 1 = 0$

We use the result in exercise 08. Notice that, $a_0 = 1$ and $a_n = 1$

so if $\frac{p}{q}$ is a root of $x^4 - 10x^2 + 1 = 0$, we have:

p must divide $1 \Rightarrow p = \pm 1$

q must divide $1 \Rightarrow q = \pm 1$

Then, the only possible rational roots of the equation $x^4 - 10x^2 + 1 = 0$ are ± 1 and these do not satisfy the equation

$\Rightarrow \sqrt{2} + \sqrt{3}$ cannot be rational number.

Exercise 10:

Let us assume $a < b$ adding a to both side : $2a < a+b \Leftrightarrow a < \frac{a+b}{2}$

In the other hand, adding b to both sides $a+b < 2b$ and $\frac{a+b}{2} < b$

Thus $a < \frac{a+b}{2} < b$.

Let us now, prove that $\frac{a+b}{2} \in \mathbb{Q}$: We take: $\begin{cases} a = \frac{p}{q} \\ b = \frac{r}{s} \end{cases}$

where $p, q, r, s \in \mathbb{Z}$

$$\Rightarrow \frac{a+b}{2} = \frac{1}{2}\left(\frac{p}{q} + \frac{r}{s}\right) = \frac{1}{2}\left(\frac{ps+qr}{qs}\right) = \frac{ps+qr}{2qs}$$

where $\begin{cases} (ps+qr) \in \mathbb{Z} \\ \text{and} \\ 2qs \in \mathbb{Z} \end{cases} \Rightarrow \frac{ps+qr}{2qs} \in \mathbb{Q}$