# Solution of series $N°1$

***Exercise 1:***

1. **a)** Let's show that: $\sup(-A) = -\inf(A)$ and $\inf(-A) = -\sup(A)$.

   we have: $\forall x \in A,\ x \geq \inf(A) \Longrightarrow -x \leq -\inf(A)$. So "$-\inf(A)$" is an upper bound of $-A$, and since $\sup(-A)$ is the smallest upper bound of $-A$ then: $\sup(-A) \leq -\inf(A) \cdots$ **(1)**.

   On the other hand: $\forall -x \in -A,\ -x \leq \sup(-A) \Longrightarrow x > -\sup(-A)$, so $-\sup(-A)$ is a lower bound of $A$, and since $\inf(A)$ is the greatest lower bound of $A$ then $\inf(A) \geq -\sup(-A)$, therefore: $-\inf(A) \leq \sup(-A) \cdots$ **(2)** .

   From **(1)** and **(2)** we get: $\sup(-A) = -\inf(A)$.

   **b)** $\inf(-A) = -\sup(A)$ :

   ▶ We have: $\forall x \in A,\ x \leq \sup(A) \Longrightarrow -x \geq -\sup(A)$, so $-\sup(A)$ is a lower bound of $-A$, since $\inf(-A)$ is the greatest lower bound of -A then $-\sup(A) \leq \inf(-A) \cdots$ **(1)**.

   On the other hand: $\forall -x \in -A,\ -x \geq \inf(-A) \Longrightarrow x \leq -\inf(-A)$, so $-\inf(-A) \geq \sup(A) \Longrightarrow \inf(-A) \leq -\sup(A) \cdots$ **(2)**.

   From **(1)** and **(2)** we get: $-\sup(A) = \inf(-A)$.

2. We show that $\sup(A) \leq \inf(A)$.

   We have $\forall a \in A,\ b \in B:\ a \leq b \Longrightarrow a \leq \inf(B)$, so $\inf(B)$ is an upper bound of $A$, but $\sup(A)$ is the smallest upper bound of $A$ then: $\sup(A) \leq \inf(B)$.

3. We show that $A \cup B$ is a bounded subset of $\mathbb{R}$: let $x \in A \cup B$, then: $x \in A$ or $x \in B$, therefore $\inf(A) \leq x \leq \sup(A)$ and $\inf(B) \leq x \leq \sup(B)$, So $\min(\inf(A), \inf(B)) \leq x \leq \max(\sup(A), \sup(B))$.

**a)** We have: $\begin{cases} \sup(A\cup B) & \leq & \max(\sup(A),\sup(B))\cdots\cdots(1) \\ \inf(A\cup B) & \geq & \min(\inf(A),\inf(B))\cdots\cdots(2) \end{cases}$

On the other hand we have: $A\subset A\cup B$ and $B\subset A\cup B$, so

$$\begin{aligned} \sup(A) &\leq \sup(A\cup B) \\ \sup(B) &\leq \sup(A\cup B) \end{aligned}$$

then: $\max(\sup(A),\sup(B)) \leq \sup(A\cup B)\cdots\cdots(1^*)$, so from (1) and $(1^*)$ we get: $\sup(A\cup B)=\max(\sup(A),\sup(B))$. In the same way we show that $\inf(A\cup B)=\min(\inf(A),\inf(B))$.

4. We show that $\sup(A+B)=\sup(A)+\sup(B)$: we have $\forall x\in A:\ \inf(A)\leq x\leq \sup(A)$, and $\forall y\in B:\ \inf(B)\leq y\leq\sup(B)$, thus: $\inf(A)+\inf(B)\leq x+y\leq \sup(A)+\sup(B)$, so $\inf(A)+\inf(B)$ is a lower bound of $A+B$, but $\inf(A+B)$ is the greatest lower bound of $A+B$, then: $\inf(A+B)\geq\inf(A)+\inf(B)\cdots\mathbf{(1)}$. and also $\sup(A+B)\leq\sup(A)+\sup(B)\cdots\mathbf{(2)}$.

On the other hand: $\forall x\in A:\ x\leq\sup(A+B)-y$, then $\sup(A+B)-y$ is an apper bound of $A$

$$\begin{aligned} \Longrightarrow \quad \sup(A) &\leq \sup(A+B)-y,\ \forall y\in B, \\ \Longrightarrow \quad y &\leq \sup(A+B)-\sup(A),\ \forall y\in B, \\ \Longrightarrow \quad \sup(B) &\leq \sup(A+B)-\sup(A), \\ \Longrightarrow \quad \sup(A)+\sup(B) &\leq \sup(A+B)\cdots(1^*). \end{aligned}$$

From **(1)** and $(1^*)$ we get: $\sup(A+B)=\sup(A)+\sup(B)$. The same to show that $\inf(A+B)=\inf(A)+\inf(B)$.

***Exercise 2:***

1. **a)** We show that if $r \in \mathbb{Q}$, and $x \notin \mathbb{Q}$, then $r + x \notin \mathbb{Q}$. we suppose that $x + r \in \mathbb{Q}$, we have: $r \in \mathbb{Q}$ so $\exists\, p,\ q \in \mathbb{Z}$ such that $r = \frac{p}{q},\ q \neq 0$.
   And $x + r \in \mathbb{Q} \Longrightarrow \exists\, p',\ q' \in \mathbb{Z}$ such that: $x + r = \frac{p'}{q'},\ q' \neq 0$.
   So: $x = \frac{p'}{q'} - \frac{p}{q} = \frac{p'q - pq'}{q'q},\ q'q \neq 0 \Longrightarrow x \in \mathbb{Q}$. This is a contradiction because $x \notin \mathbb{Q}$, then $x + r \notin \mathbb{Q}$.

   **b)** We show that if $x \notin \mathbb{Q}$ and $r \in \mathbb{Q}$ then $x.r \notin \mathbb{Q}$:

   We have $r \in \mathbb{Q} \Longrightarrow r = \frac{p}{q},\ q \neq 0$, and $p \neq 0$ $(r \neq 0)$. We assume that $x.r \in \mathbb{Q}$, then $x.r = \frac{p'}{q'},\ q' \neq 0 \Longrightarrow x = \frac{p'}{q'}.\frac{q}{p} = \frac{p'q}{q'p},\ q'p \neq 0$, thus $x \in \mathbb{Q}$. Contradiction, then $x.r \notin \mathbb{Q}$.

2. We show that $\sqrt{2} \notin \mathbb{Q}$. Suppose that $\sqrt{2} \in \mathbb{Q} \Longrightarrow \exists\, p.\ q \in \mathbb{Z}$ such that $\sqrt{2} = \frac{p}{q},\ q \neq 0$.
   suppose that $p$ and $q$ are prime, then $\sqrt{2} = \frac{p}{q} \Longrightarrow q\sqrt{2} = p \Longrightarrow 2q^2 = p^2$, therefore $p^2$ is even $\Longrightarrow p$ is even, then $p = 2p',\ p' \in \mathbb{Z}$.

   So $2q^2 = (2p')^2 = 4p'^2 \Longrightarrow q^2 = 2p'^2$, therefore $q^2$ is even $\Longrightarrow q$ is even. Contradiction, then $\sqrt{2} \notin \mathbb{Q}$.

3. We show that $\frac{\ln 3}{\ln 2}$ is irrational. Assume that $\frac{\ln 3}{\ln 2} \in \mathbb{Q} \Longrightarrow \exists\, p,\ q \in \mathbb{Z},\ q \neq 0$ such that $\frac{\ln 3}{\ln 2} = \frac{p}{q} \Longrightarrow q \ln 3 = p \ln 2 \Longrightarrow e^{q \ln 3} = e^{p \ln 2} \Longrightarrow 3^q = 2^p$.

   - If $p = 0$, then $3^q = 2^0 = 1 \Longrightarrow q = 0$ (contradiction because $q \neq 0$).
   - If $p > 0$, then $3^q$ is odd and $2^p$ is even. (contradiction), then $\frac{\ln 3}{\ln 2} \notin \mathbb{Q}$.

***Exercise 3:***

1. We show that if $A$ is bounded then $B$ is bounded. $A$ is bounded $\Longleftrightarrow \exists\, m,\ M \in \mathbb{R},\ \forall x \in A:\ m \leq x \leq M$.

We have $B \subset A \Longleftrightarrow \forall x \in B,\ x \in A$, and $A$ is bounded so $m \leq x \leq M$, then $B$ is bounded.

2. **a)** Show that $\inf(A) \leq \inf(B)$. We have $B \subset A \Longrightarrow \forall x \in B:\ x \geq \inf(A)$, therefore $\inf(A)$ is an upper bound of $B$, then $\inf(A) \leq \inf(B)$ because $\inf(B)$ is the greatest upper bound of $B$.

   **b)** Show that $\sup(A) \geq \sup(B)$. We have $B \subset A$, then $\forall x \in B:\ \inf(A) \leq x \leq \sup(A)$, therefore $\sup(A)$ is an upper bound of $B$, and since $\sup(B)$ is the smallest upper bound of $B$ then $\sup(B) \leq \sup(A)$.

***Exercise 4:***

1. $A = \left\{ a_n \in \mathbb{R} \mid a_n = \dfrac{n+3}{\frac{n}{4}+1},\ n \in \mathbb{N} \right\}$. We show that $A$ is bounded, i.e: $\exists\, m,\ M \in \mathbb{R} |\ \forall a_n \in A:\ m \leq a_n \leq M$. We have: $\forall n \in \mathbb{N}$

$$\begin{aligned} \frac{n+3}{\frac{n}{4}+1} &= 4\left(\frac{n+3}{n+4}\right) \\ &= 4\left(\frac{n+4-1}{n+4}\right) \\ &= 4\left(1 - \frac{1}{n+4}\right) = 4 - \frac{4}{n+4} \end{aligned}$$

$\forall n \geq 0,\ n+4 \geq 4 \Longrightarrow \dfrac{1}{n+4} \leq \dfrac{1}{4}$, therfore $-\dfrac{4}{n+4} \geq -1 \Longrightarrow 4 - \dfrac{4}{n+4} \geq 3 \Longrightarrow a_n \geq 3 \cdots\cdots$ **(1)**.

$\forall n \geq 0:\ n+4 \geq 4 > 0 \Longrightarrow \dfrac{1}{n+4} > 0$, so $-\dfrac{4}{n+4} < 0 \Longrightarrow 4 - \dfrac{4}{n+4} < 4 \Longrightarrow a_n < 4 \cdots\cdots$ **(2)**.

Then from **(1)** and **(2)**, we get $3 \leq a_n \leq 4$. So $\inf(A) = 3$, and since $3 \in A$, then: $\inf(A) = \min(A) = 3,\ (a_0 = 3 \in A)$, and $\sup(A) = 4$.

Now let's show that $\sup(A) = 4$.

$$\sup(A) = 4 \Longleftrightarrow \begin{cases} \forall\ a_n \in A:\ a_n < 4, \\ \forall \varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N}:\ a_n > 4 - \varepsilon. \end{cases}$$

We have: $a_n < 4,\ \forall a_n \in A$ verify: $\forall\ \varepsilon > 0,\ a_n > 4 - \varepsilon \Longrightarrow 4 - \dfrac{4}{n+4} > 4 - \varepsilon \Longrightarrow \dfrac{4}{n+4} < \varepsilon$, therfore: $\dfrac{n+4}{4} > \dfrac{1}{\varepsilon} \Longrightarrow n + 4 > \dfrac{4}{\varepsilon} \Longrightarrow n > \dfrac{4}{\varepsilon} - 4$. Just take $n_\varepsilon = \left[\dfrac{4}{\varepsilon} - 4\right] + 1$, then $\sup(A) = 4$.

2. $B = \left\{b_n \in \mathbb{R} \mid b_n = \dfrac{1}{n^2} + \dfrac{2}{n} + 4\right\}$. We show that $B$ is bounded, for all $n \geq 1 \Longrightarrow \dfrac{2}{n} \leq 2$, and $\dfrac{1}{n^2} \leq 1$, therfore $\dfrac{2}{n} + \dfrac{1}{n^2} \leq 3 \Longrightarrow \dfrac{2}{n} + \dfrac{1}{n^2} + 4 \leq 7$, then $b_n \leq 7 \cdots\cdots$ **(1)**.

   On the other hand: $\dfrac{2}{n} > 0$, and $\dfrac{1}{n^2} > 0$, then $\dfrac{2}{n} + \dfrac{1}{n^2} > 0 \Longrightarrow \dfrac{2}{n} + \dfrac{1}{n^2} + 4 > 4$, so $b_n > 4 \cdots\cdots$ **(2)**.

   From **(1)**, and **(2)**, we get: $\forall n \in \mathbb{N},\ 4 < b_n \leq 7$, then $B$ is bounded in $\mathbb{R}$, such that $\sup(B) = \max(B) = 7$, and $\inf(B) = 4$. Now we must to prove that $\inf(B) = 4$.

$$\inf(B) = 4 \Longleftrightarrow \begin{cases} \forall\ b_n \in B,\ b_n > 4, \\ \forall\ \varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N}^* :\ b_n < 4 + \varepsilon. \end{cases}$$

   We have $b_n < 4 + \varepsilon \Longrightarrow \dfrac{1}{n^2} + \dfrac{2}{n} + 4 < 4 + \varepsilon \Longrightarrow \dfrac{1}{n^2} + \dfrac{2}{n} < \varepsilon$, also: $n^2 \geq n \Longrightarrow \dfrac{1}{n^2} \leq \dfrac{1}{n}$, and $\dfrac{1}{n^2} + \dfrac{2}{n} \leq \dfrac{3}{n}$.

   We are only looking for a $n_\varepsilon$ such that $\dfrac{3}{n} < \varepsilon$, i.e, $n > \dfrac{3}{\varepsilon}$, therfore we just take $n_\varepsilon = \left[\dfrac{3}{\varepsilon}\right] + 1$, then $\inf(B) = 4 = \sup(A)$.

***Exercise 5:***

1. $A = \{ax + b \mid x \in [-2, 1],\ a,\ b \in \mathbb{R}\}$. Assume that:

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow f(x) = ax + b \end{aligned}$$

   - If $a = 0 \Longrightarrow f(x) = b$, then $f$ is constant, and $A = \{b\}$ is bounded such that $\sup(A) = \inf(A) = b$.

- If $a > 0 \Longrightarrow f(x)$ is increasing, so for all $-2 \leq x \leq 1$, we have:

  $f(-2) \leq f(x) \leq f(1) \Longrightarrow -2a + b \leq f(x) \leq a + b$, then $\forall x \in [-2, 1]$, $A$ is bounded such that: $\inf(A) = \min(A) = -2a + b$, and $\sup(A) = \max(A) = a + b$.

- If $a < 0 \Longrightarrow f(x)$ is decreasing, then $A$ is bounded and $\inf(A) = a + b$, $\sup(A) = -2a + b$.

2. $B = \left\{2 - \dfrac{1}{n},\ n \in \mathbb{N}^*\right\}$. For $n = 1 \Longrightarrow B = 1$, and for $n \longrightarrow \infty \Longrightarrow B = 2$, so $B = [1, 2[$. The set of upper bounds of $B$ is $[2, +\infty[$ therefore $\sup(B) = 2$, and the set of lower bounds of $B$ is $]-\infty, 1]$, therfore $\inf(B) = 1$, since $1 \in B$, then $\inf(B) = \min(B) = 1$, and $\max(B)$ does not exist because $2 \notin B$.

***Exercise 6:***

1. $z_1 = \dfrac{5 + 2i}{1 - 2i} = \dfrac{(5 + 2i)(1 + 2i)}{(1 - 2i)(1 + 2i)} = \dfrac{1 + 12i}{5} = \dfrac{1}{5} + \dfrac{12}{5}i.$

   $z_2 = \dfrac{-2}{1 - i\sqrt{3}} = \dfrac{-2(1 + i\sqrt{3})}{1^2 + (-\sqrt{3})^2} = \dfrac{-2(1 + i\sqrt{3})}{4} = \dfrac{-1}{2} - i\dfrac{\sqrt{3}}{2}.$

2. Solve: $z^4 + (3-6i)z^2 - 8 - 6i = 0$. We suppose $x = z^2$, so $x^2 + (3-6i)x - 8 - 6i = 0$.

   $\triangle = (3 - 6i)^2 - 4(-8 - 6i) = 5 - 12i$, the square roots of $5 - 12i$ are:

   $(a+ib)^2 = 5-12i \Longleftrightarrow a^2 - b^2 + 2iab = 5 - 12i \Longleftrightarrow \begin{cases} a^2 - b^2 = 5 \cdots L_1 \\ 2ab = -12 \Rightarrow ab = -6 \cdots L_2 \end{cases}$

   We add the equality of the modules

   $$a^2 + b^2 = \sqrt{5^2 + 12^2} = \sqrt{169} = 13 \cdots L_3$$

   $L_1 + L_3 \Longleftrightarrow 2a^2 = 18$, hence $a^2 = 9 \Longleftrightarrow a = \pm 3$, and

   $L_1 - L_3 \Longleftrightarrow 2b^2 = 8$, hence $b^2 = 4 \Longleftrightarrow b = \pm 2$.

According to $L_2$: $a$ and $b$ have the same sign so the two square roots of $5-12i$ are: $3+2i$, and $-3-2i$.

The solutions of $x^2+(3+6i)x-8-6i$ are:

$$\begin{aligned} x_1 &= \frac{-(3-6i)-(3+2i)}{2} &= -3+4i \\ x_2 &= \frac{-(3-6i)+(3+2i)}{2} &= 2i \end{aligned}$$

- $x_1 = -3+4i = -4+4i+1 = -(2-i)^2$, so $z^2 = -3+4i$ has two solutions: $z_1 = -2+i$, and $z_2 = -2-i$.
- $x_2 = 2i = (1+i)^2$, so $z^2 = 2i$ has two solutions: $z_3 = 1+i$, and $z_4 = -1-i$.

***Exercise 7:***

Calculate $\cos 5\theta$, and $\sin 5\theta$. We have by the Moivre's formula:

$$\cos 5\theta + i\sin 5\theta = e^{i5\theta} = (e^{i\theta})^5 = (\cos\theta + i\sin\theta)^5.$$

Using Newton's binomial formula:

$$(\cos\theta + i\sin\theta)^5 =$$

$$\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

So: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$,

and $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

***Exercise 8:***

1. **a)** $|u| = \left|\dfrac{\sqrt{6}-i\sqrt{2}}{2}\right| = \dfrac{\sqrt{6+2}}{2} = \dfrac{\sqrt{8}}{2} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}.$

   $u = \dfrac{\sqrt{6}-i\sqrt{2}}{2} = \sqrt{2}\left(\dfrac{\sqrt{2}\times\sqrt{3}-i\sqrt{2}}{2\sqrt{2}}\right) = \sqrt{2}\left(\dfrac{\sqrt{3}-i}{2}\right) = \sqrt{2}\,e^{-i\frac{\pi}{6}}.$

   Then $|u| = \sqrt{2}$, and $arg(u) = -\dfrac{\pi}{6}$.

   **b)** $|v| = \sqrt{1^2+(-1)^2} = \sqrt{2}.$

$$v = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2}\ e^{-i\frac{\pi}{4}}.$$

Then $|v| = \sqrt{2}$, and $arg(v) = -\frac{\pi}{4}$.

2. $\frac{u}{v} = \frac{\sqrt{2}\ e^{-i\frac{\pi}{6}}}{\sqrt{2}\ e^{-i\frac{\pi}{4}}} = e^{i(-\frac{\pi}{6}+\frac{\pi}{4})} = e^{i\frac{\pi}{12}}.$

Then: $\left|\frac{u}{v}\right| = 1$, and $arg(\frac{u}{v} = \frac{\pi}{12})$.