
Exercises Serie N° 2

Exercise 1

Show by induction that:

1. $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$.
2. $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 2

Determine, by justifying your answers, if the following sequences are convergent:

1. $U_n = \frac{\cos n - 2}{n^4}, \forall n \in \mathbb{N}^*$.
2. $V_n = \frac{3n + 5(-1)^n}{2n + 1}, \forall n \in \mathbb{N}$.
3. $W_n = (-1)^n \left(\frac{n+1}{n}\right), \forall n \in \mathbb{N}^*$.
4. $Z_n = \sqrt{2n+1} - \sqrt{2n-1}, \forall n \in \mathbb{N}^*.(*)$

Exercise 3

Let $(u_n)_{n \in \mathbb{N}}$ be the sequence of real numbers defined by $u_0 \in]0,1]$, and by the recurrence relation

$$u_{n+1} = \frac{u_n}{2} + \frac{(u_n)^2}{4}$$

1. Show that: $\forall n \in \mathbb{N}, u_n > 0$.
2. Show that: $\forall n \in \mathbb{N}, u_n \leq 1$.
3. Show that the sequence is monotonic. Deduce that the sequence is convergent.
4. Determine the limit of the sequence $(u_n)_{n \in \mathbb{N}}$.

Exercise 4

Prove that the following two sequences are adjacent

$$\forall n \in \mathbb{N}, \quad u_n = \sum_{k=1}^n \frac{1}{k^2}, \quad v_n = u_n + \frac{1}{n}.$$

Exercise 5

1. Let $u_n = \frac{E(\sqrt{n})}{n}$, for all $n \in \mathbb{N}^*$, show that

$$\lim_{n \rightarrow +\infty} u_n = 0.$$

2. Let $v_n = \frac{E(\sqrt{n})^2}{n}$, for all $n \in \mathbb{N}^*$, show that the sequence $(v_n)_{n \in \mathbb{N}^*}$ converges and determine its limit. (*)

Exercise 6

Calculate the following limits, if they exist, of the following sequences:

1. $u_n = \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{(n+1)(n+2)}$.

2. $v_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}$.

3. $w_n = \frac{\ln(n+1)}{\ln n}$.

4. $z_n = \sqrt{n^2 + n + 1} - \sqrt{n^2 - n + 1}$. (*)

Exercise 7

We consider the sequence $(u_n)_{n \geq 1}$ given by: $u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$.

1. Show that $\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$.

2. Show that the sequence $(u_n)_{n \geq 1}$ is bounded above by 2.

3. Show that the sequence $(u_n)_{n \geq 1}$ is increasing.

4. Deduce that $(u_n)_{n \geq 1}$ is converges.

Exercise 8

We consider the sequence $(u_n)_{n \in \mathbb{N}}$ defined by $u_0 = 0$ and by the recurrence relation

$$u_{n+1} = \frac{1}{6}u_n^2 + \frac{3}{2}$$

1. Show that for all $n \in \mathbb{N}^*$, $u_n > 0$.

2. Calculate the limit of the sequence $(u_n)_{n \in \mathbb{N}}$.

3. Show that for all $n \in \mathbb{N}$, $u_n < 3$.

4. Show that the sequence is increasing, what can we conclude from this?

Exercise 9 (Supplementary)

We consider the sequence $(u_n)_{n \in \mathbb{N}^*}$ defined by

$$u_n = \frac{1}{3 + |\sin(1)|\sqrt{1}} + \frac{1}{3 + |\sin(2)|\sqrt{2}} + \cdots + \frac{1}{3 + |\sin(n)|\sqrt{n}}$$

Show that $\lim_{n \rightarrow +\infty} u_n = +\infty$.

Exercises marked with (*) are left to students.