# *CHAPTER 2*

# *The Numerical Sequences*

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# CHAPTER 1

# THE NUMERICAL SEQUENCES

## 1.1 The general concept of a sequence

### 1.1.1 *Definition*

**Definition 1.1.1.** *A sequence $u$ or $(u_n)_{n\in\mathbb{N}}$ is a function from $\mathbb{N}$ to $\mathbb{R}$:*

$$\begin{aligned} u:\ \ \mathbb{N} &\longrightarrow \mathbb{R}, \\ n &\longmapsto u_n. \end{aligned}$$

The elements of a sequence are called **the terms**. The $n-th$ term of a sequence is called **the general term** of the sequence, and we denote $(u_n)_{n\in\mathbb{N}}$.

**Example 1.1.1.** *1.* $(\sqrt{n})_{n\geq 0}$ *is the sequence of terms:* $0, 1, \sqrt{2}, \sqrt{3}, \cdots$.

*2.* $((-1)^n)_{n\geq 0}$ *is the sequence of terms that are alternated:* $+1, -1, +1, -1, \cdots$.

### 1.1.2 *Explicit definition*

Just as real-valued functions from Calculus one were usually expressed by a formula, we will most often encounter sequences that can be expressed by a formula. For example: $u_n = 3n + 1$, $v_n = \sin(n\pi/6)$, $w_n = (1/2)^n$.

### 1.1.3 *Definition by recurrence*

Another way to define a sequence is recursive, that is, by giving the first term and a recurrence relation between a term and the next term. For example

$$\begin{cases} u_0 = 1, \\ u_{n+1} = 2u_n + 3, \ n \in \mathbb{N}. \end{cases}$$

## 1.2 Qualitative features of sequences

### 1.2.1 *Monotonicity*

**Definition 1.2.1.** *A sequence $(u_n)_{n\in\mathbb{N}}$ is called* ***increasing*** *(or* ***strictly increasing****) if $u_n \leq u_{n+1}$ (or $u_n < u_{n+1}$), for all $n \in \mathbb{N}$.*
*Similarly a sequence $(u_n)_{n\in\mathbb{N}}$ is* ***decreasing*** *(or* ***strictly decreasing****) if $u_n \geq u_{n+1}$ (or $u_n > u_{n+1}$), for all $n \in \mathbb{N}$.*
*If a sequence is increasing (or strictly increasing), decreasing (or strictly decreasing ), it is said to be* ***monotonic*** *(or* ***strictly monotonic****).*

**Example 1.2.1.** *The sequence $u_n = \dfrac{2^n - 1}{2^n}$ which starts*

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$$

*is increasing. On the other hand, the sequence $v_n = \dfrac{n+1}{n}$ which starts*

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

*is decreasing.*

### 1.2.2 *Boundedness*

**Definition 1.2.2.** *A sequence $(u_n)_{n\in\mathbb{N}}$ is* ***bounded above*** *if there is some number $M$ so that for all $n \in \mathbb{N}$, we have $u_n \leq M$. Likewise, a sequence $(u_n)_{n\in\mathbb{N}}$ is* ***bounded below*** *if there is some number $m$ so that for every $n \in \mathbb{N}$, we have $u_n \geq m$.*
*If a sequence is both bounded above and below, the sequence is said to be* ***bounded****.*

**Remark 1.2.1.** *If a sequence $\{u_n\}_{n=0}^{\infty}$ is increasing it is bounded below by $u_0$, and if it is decreasing it is bounded above by $u_0$.*

**Theorem 1.2.1.** *If the sequence $(u_n)$ is bounded and monotonic, then $\lim_{n\to\infty} u_n$ exists.*

## 1.3 Convergent Sequences

**Definition 1.3.1.** *We say that the sequence $(u_n)_{n\in\mathbb{N}}$ is converge to the scalar $l$ if*

$$\forall \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} :\ \forall n \geq n_0 :\ |u_n - l| < \varepsilon.$$

*In this case we write $\lim_{n\to\infty} u_n = l$, ($l$ finite). If there is no a finite value $l$ such that $\lim_{n\to\infty} u_n = l$, then we say that the limit does not exist, or equivalently that the sequence diverges.*

**Remark 1.3.1.** *Any open interval with center $l$ contains all the terms of the sequence from a certain rank.*

**Example 1.3.1.**

1. $u_n = \left(\frac{3}{4}\right)^n$.

   $$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \left(\frac{3}{4}\right)^n = \lim_{n\to+\infty} e^{n\ln\left(\frac{3}{4}\right)} = 0.$$ *So $(u_n)$ converges to* 0.

2. $v_n = (-1)^n$. *$v_n$ is a divergent sequence.*

3. $w_n = \sin(n)$. *The limit of $w_n$ does not exist, so $w_n$ is divergent.*

**Proposition 1.3.1.** *If the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent then it has a unique limit.*

**Proof 1.** *Assume that $\lim_{n\to+\infty} u_n = l$, and $\lim_{n\to+\infty} u_n = l'$, we need to show that $l = l'$.*

- $\lim_{n\to+\infty} u_n = l \iff \forall\varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} : \forall\, n \geq n_0 :\ |u_n - l| < \dfrac{\varepsilon}{2}.$
  *and*
- $\lim_{n\to+\infty} u_n = l' \iff \forall\varepsilon > 0,\ \exists\, n_1 \in \mathbb{N} : \forall\, n \geq n_1 :\ |u_n - l'| < \dfrac{\varepsilon}{2}.$

*We have* $|l - l'| = |l - u_n + u_n - l'| \leq |l - u_n| + |u_n - l'| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. *So* $\forall\varepsilon > 0 :\ |l - l'| < \varepsilon$, *then* $l = l'$.

**Proposition 1.3.2.** *If the sequence* $(u_n)_{n\in\mathbb{N}}$ *converges to l, then the sequence* $|u_n|$ *converges to* $|l|$.

**Proposition 1.3.3.** *any convergent sequence is bounded.*

**Remark 1.3.2.**
- *If* $(u_n)_{n\in\mathbb{N}}$ *is increasing and bounded above, then* $(u_n)_{n\in\mathbb{N}}$ *converges to* $l = \sup u_n$.
- *If* $(u_n)_{n\in\mathbb{N}}$ *is decreasing and bounded below, then* $(u_n)_{n\in\mathbb{N}}$ *is converges to* $l = \inf u_n$.

## 1.4 The usual rules of limits

If $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are convergent sequences to $l$ and $l'$ respectively, and $\alpha$ is any real constant then:

1) $\lim_{n\to+\infty}(u_n + v_n) = l + l'$,
2) $\lim_{n\to+\infty}(u_n \times v_n) = l \times l'$,
3) $\lim_{n\to+\infty}(\alpha u_n) = \alpha l$,
5) $\lim_{n\to+\infty} \dfrac{1}{u_n} = \dfrac{1}{l},\ l \neq 0$,
6) $if\ u_n \leq v_n,\ then\ l \leq l'$,
7) $if\, l = l',\ and\ u_n \leq w_n \leq v_n,\ then\ \lim_{n\to+\infty} w_n = l.$

## 1.5 Adjacent sequences

**Definition 1.5.1.** *We say that two real sequences* $(u_n)_{n\in\mathbb{N}}$, *and* $(v_n)_{n\in\mathbb{N}}$ *are adjacent if they satisfy the following properties:*

1. $(u_n)_{n\in\mathbb{N}}$ *is increasing, and* $(v_n)_{n\in\mathbb{N}}$ *is decreasing,*
2. $\lim_{n\to\infty}(u_n - v_n) = 0.$

**Theorem 1.5.1.** *If the two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are adjacent then they converge to the same limit.*

*Proof.* We assume that $(u_n)_{n\in\mathbb{N}}$ is increasing and $(v_n)_{n\in\mathbb{N}}$ is decreasing. Let $w_n = u_n - v_n$, then

$$\begin{aligned} w_{n+1} - w_n &= u_{n+1} - v_{n+1} - u_n + v_n, \\ &= (u_n + 1 - u_n) - (v_{n+1} - v_1), \\ &\geq 0. \end{aligned}$$

and $\lim_{n\to\infty} w_n = \lim_{n\to\infty}(u_n - v_n) = 0$. Since $(w_n)$ is an increasing sequence and $\lim_{n\to\infty} w_n = 0$, then $\forall n \in \mathbb{N}: \ w_n \leq 0 \Rightarrow u_n \leq v_n$.
Therefore, $\forall n \in \mathbb{N}: \ u_0 \leq u_n \leq v_n \leq v_0$. the sequence $(u_n)$ is convergent since it is increasing and bounded above by $v_0$, also the sequence $(v_n)$ is convergent, and since $\lim_{n\to\infty}(u_n - v_n) = 0$ we deduce that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$. □

**Exercise 1.5.1.** *Show that the two sequences $(u_n)$ and $(v_n)$ are adjacent:*

- $u_n = 1 + \frac{1}{n!}$, *and* $v_n = \frac{n}{n+1}$.
- $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ *and* $v_n = u_n + \frac{2}{n+1}$.

## 1.6 Subsequences

It is useful to sometimes consider only some terms of a sequence. A subsequence of $\{u_n\}_{n=1}^{\infty}$ is a sequence that contains only some of the numbers from $\{u_n\}_{n=1}^{\infty}$ in the same order.

**Definition 1.6.1.** *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence. Let $\{n_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of natural numbers, that is $n_i < n_{i+1}$ for all $i \in \mathbb{N}$ (in other words, $n_1 < n_2 < n_3 < \cdots$). The sequence $\{u_{n_i}\}_{i=1}^{\infty}$ is called a subsequence of $\{u_n\}_{n=1}^{\infty}$.*
*So the subsequence is the sequence $u_{n_1}, \ u_{n_2}, \ u_{n_3}, \ \cdots$.*

**Example 1.6.1.** *Consider the sequence*

$$u_n = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\},$$

*then letting $n_k = 2k$ yields the subsequence*

$$u_{2k} = \left(\frac{1}{2k}\right)_{k=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2k}, \cdots\right\},$$

*and letting $n_k = 2k+1$ yields the subsequence*

$$u_{2k+1} = \left(\frac{1}{2k+1}\right)_{k=1}^{\infty} = \left\{\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2k+1}, \cdots\right\}.$$

**Proposition 1.6.1.** *If $\{u_n\}_{n=1}^{\infty}$ is a convergent sequence, then every subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ is also convergent, and*

$$\lim_{n\to+\infty} u_n = \lim_{i\to+\infty} u_{n_i}.$$

**Corollary 1.6.1.** *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence, if it admits a divergent subsequence, or if it admits two subsequences converging to distinct limits, then it diverges.*

**Theorem 1.6.1.** ***(Bolzano-Weierstrass)***
*Every bounded sequence has a convergent subsequence.*

## 1.7 Cauchy Sequences

**Definition 1.7.1.** *A real sequence $(u_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$, if $m, n \geq N$ then*

$$|u_n - u_m| \leq \varepsilon.$$

**Proposition 1.7.1.** *If a sequence is Cauchy, then it is bounded.*

**Proposition 1.7.2.** *A sequence of real numbers is Cauchy if and only if it converges.*

## 1.8 Arithmetic sequences

### 1.8.1 *Definition*

A simple way to generate a sequence is to start with a number $u_0$, and add to it a fixed constant $r$, over and over again. This type of sequence is called an **arithmetic sequence**.

**Definition 1.8.1.** *the sequence* $(u_n)_{n\in\mathbb{N}}$ *is an arithmetic sequence with first term* $u_0$ *and common difference* $r$ *if and only if for any integer* $n \in \mathbb{N}$ *we have*

$$u_{n+1} = u_n + r, \qquad (u_n = u_0 + n.r).$$

*More generally:* $u_n = u_p + (n-p).r$.

### 1.8.2 *Sum of* $n$ *terms*

For the arithmetic sequence

$$S_n = u_0 + u_1 + \cdots + u_{n-1} = n.\frac{u_0 + u_{n-1}}{2}.$$

## 1.9 Geometric sequences

### 1.9.1 *Definition*

Another simple way of generating a sequence is to start with a number $v_0$ and repeatedly multiply it by a fixed nonzero constant $q$. This type of sequence is called a geometric sequence.

**Definition 1.9.1.** *the sequence* $(v_n)_{n\in\mathbb{N}}$ *is a geometric sequence with first term* $v_0$ *and common ratio* $q \in \mathbb{R}^*$ *if and only if for any integer* $n \in \mathbb{N}$ *we have*

$$v_{n+1} = q.v_n, \qquad (v_n = v_0.q^n).$$

*More generally:* $v_n = v_p.q^{n-p}$.

### 1.9.2 *Sum of* $n$ *terms*

For a geometric sequence, if $S_n = 1 + q + q^2 + \cdots + q^n$, then

$$S_n = \begin{cases} n+1 & si \quad q = 1, \\ \dfrac{1-q^{n+1}}{1-q} & si \quad q \neq 1. \end{cases}$$