CHAPTER 1+2

Real Numbers and Complex Numbers

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CHAPTER 1

FIELD OF REAL NUMBERS

We recall the usual notation for sets of numbers:

$\mathbb{N} = \{0, 1, 2, \cdot, n, \cdots\}$: is the set of natural numbers.

$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$: is the set of relative integers.

$\mathbb{Q} = \{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N}^*\}$: is the set of rationals.

$\mathbb{D} = \{r = \frac{p}{10^k} \in \mathbb{Q},\ p \in \mathbb{Z},\ k \in \mathbb{N}\}$: is the set of decimal numbers.

$\mathbb{R}$: the set of real numbers.

These sets are subsets of one another in the order $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{D} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, where each set is a proper subset of the next.

1.1 Properties of the real numbers

The set of real numbers is a set on which we must define two internal operations denoted + and ., and a total order relationship denoted $\leq$ $(or\ \geq)$.

The two operations (addition and multiplication) are defined by:

$$\forall x,\ y \in \mathbb{R}: \quad (x, y) \longrightarrow x + y \quad (Addition)$$
$$\forall x,\ y \in \mathbb{R}: \quad (x, y) \longrightarrow x \times y \quad (Multiplication)$$

These operations verify the following axioms:

1. **Axiom 1:** $\mathbb{R}$ is a commutative field. For all $x,\ y,\ z \in \mathbb{R}$
   - $(x + y) + z = x + (y + z)$ (Associative Law for Addition).
   - $x + y = y + x$ (Commutative Law for Addition).
   - $x + 0 = 0$ (Identity Law for Addition).
   - $x + (-x) = 0$ (Inverses Law for Addition).
   - $(xy)z = x(yz)$ (Associative Law for Multiplication).
   - $xy = yx$ (Commutative Law for Multiplication).
   - $x.1 = x$ (Identity Law for Multiplication).
   - If $x \neq 0$, then $xx^{-1} = 1$ (Inverses Law for Multiplication).
   - $x(y + z) = xy + xz$ (Distributive Law).
2. **Axiom 2:** $\mathbb{R}$ is a totally ordered field. For all $x,\ y,\ z \in \mathbb{R}$
   - $x \leq x$ (Reflexive Law).
   - If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetric Law).
   - If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitive Law).
   - If $x \leq y$, then $x + z \leq y + z$ (Addition Law for Order).
   - If $x \leq y$ and $z > 0$, then $xz \leq yz$ (Multiplication Law for Order).
3. **Axiom 3:**
   - Any nonempty subset A of $\mathbb{R}$ and bounded above admits an upper bound that we denote by $\sup(A)$.
   - Any nonempty subset A of $\mathbb{R}$ and bounded below admits a lower bound that we denote by $\inf(A)$.

Remark 1.1.1. *Let A be a non-empty subset of* $\mathbb{R}$*, then:*

- $A = \{x \in \mathbb{R} \mid x \in A\}$.
- $-A = \{x \in \mathbb{R} \mid -x \in A\}$.

Proposition 1.1.1. ***Newton's binomial formula***
Let $x,\ y \in \mathbb{R}$*, and* $n \in \mathbb{N}^*$*, we have*

$$(x+y)^n = \sum_{k=0}^{n} C_n^k\, x^k\, y^{n-k},\ \textit{where } C_n^k = \frac{n!}{k!(n-k)!},\ 1! = 1 \textit{ and } 0! = 1.$$

1.2 Intervals

We now define various types of intervals in the real numbers. you most likely encountered the use of intervals in previous mathematics courses, for example, precalculus and calculus, but their importance might not have been evident in those courses. By contrast, the various types of intervals play a fundamental role in real analysis.

Definition 1.2.1. *An* ***open bounded interval*** *is a set of the form*

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$*. A* ***closed bounded interval*** *is a set of the form*

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$*. A* ***half-open interval*** *is a set of the form*

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}, \qquad or \qquad (a,b] = \{x \in \mathbb{R} \mid a < x \leq b\},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$*. An* ***open unbounded interval*** *is a set of the form*

$$(a,\infty) = \{x \in \mathbb{R} \mid a < x\}, \qquad or \qquad (-\infty,b) = \{x \in \mathbb{R} \mid x < b\}, \qquad or \qquad (-\infty,\infty) = \mathbb{R},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$*. A* ***closed unbounded interval*** *is a set of the form*

$$[a,\infty) = \{x \in \mathbb{R} \mid a \leq x\}, \qquad or \qquad (-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\},$$

where $a,\ b \in \mathbb{R}$ and $a \leq b$.

Notation:

$\mathbb{R}_+^* = \{x \in \mathbb{R},\ x > 0\}$, $\mathbb{R}_-^* = \{x \in \mathbb{R},\ x < 0\}$, $\mathbb{R}^* = \mathbb{R} - \{0\}$,

$\mathbb{R}_+ = \{x \in \mathbb{R},\ x \geq 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R},\ x \leq 0\}$.

1.3 Completed number line $\overline{\mathbb{R}}$: (Extension of $\mathbb{R}$)

Definition 1.3.1. *We denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty,\ +\infty\}$. This set is called a completed number line.*

Order relation on $\mathbb{R}$

$\mathbb{R}$ is provided with a total order $\leq$ extending that of $\mathbb{R}$ and further defined by:

$$\forall x \in \mathbb{R},\ -\infty \leq x \leq +\infty, \qquad \text{(in fact } -\infty < x < +\infty \text{).}$$

operations on $\overline{\mathbb{R}}$

Similarly, the laws + and . of $\mathbb{R}$ are "extended" (always commutatively) by posing

1) $(+\infty) + (+\infty) = (+\infty)$; $(-\infty) + (-\infty) = (-\infty)$.
2) $\forall x \in \mathbb{R},\ x + (+\infty) = +\infty$; $x + (-\infty) = -\infty$.
3) $(+\infty)(+\infty) = +\infty$; $(-\infty)(-\infty) = +\infty$; $(+\infty)(-\infty) = -\infty$.
4) $\forall x \in \mathbb{R}_-^*,\ x(+\infty) = -\infty$; $x(-\infty) = +\infty$.
5) $\forall x \in \mathbb{R}_+^*,\ x(+\infty) = +\infty$; $x(-\infty) = -\infty$.

Indeterminate forms

The following expressions are called indeterminate forms:

$$(+\infty) + (-\infty);\ 0(-\infty);\ 0(+\infty);\ \frac{\infty}{\infty};\ \frac{0}{0};\ 1^{\infty},\ 0^0,\ \infty^0.$$

1.4 Archimedes property

The set $\mathbb{R}$ verifies Archimedes' axiom, i.e.

$$\forall x \in \mathbb{R},\ \exists\, n \in \mathbb{N} \text{ such that}:\ n > x$$

In other words the set $\mathbb{N}$ is not bounded in $\mathbb{R}$.

1.5 Rational and irrational numbers

Definition 1.5.1. *The set of rational numbers, denoted $\mathbb{Q}$, is defined by*

$$\mathbb{Q} = \left\{x \in \mathbb{R} \mid x = \tfrac{p}{q}\ for\ some\ p,\ q \in \mathbb{Z}\ such\ that\ q \neq 0\right\}$$

The elements of $\mathbb{R}|\mathbb{Q}$ are called irrational numbers.

1.6 The density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.6.1. *Between any two reals there exists a rational, that is to say*

$$\forall x, y \in \mathbb{R},\ x < y \Longrightarrow \exists\, q \in \mathbb{Q} \text{ such that } x < q < y$$

1.7 Absolute value

Definition 1.7.1. *Let $x \in \mathbb{R}$, we call the absolute value of x denoted $|x|$ the real defined by:*

$$|x| = \begin{cases} x & : x > 0 \\ 0 & : x = 0 \\ -x & : x < 0 \end{cases}$$

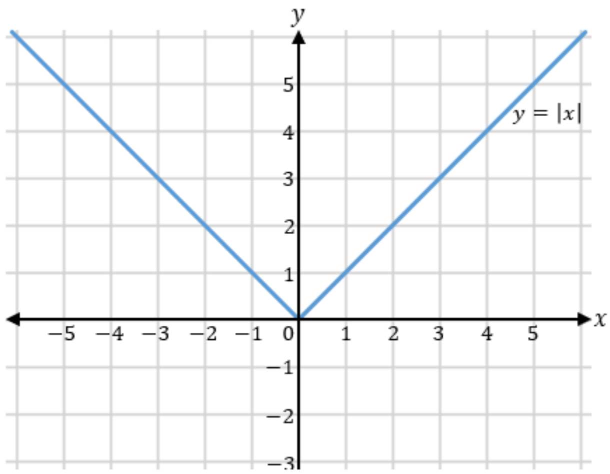

Figure 1.1: graphical presentation of $y = |x|$

Absolute Value Properties

The absolute value verify the following properties:

1. $\forall x \in \mathbb{R} : \ |x| \geq 0$
2. $\forall x, \ y \in \mathbb{R} : \ |x.y| = |x|.|y|$
3. $\forall x, \ y \in \mathbb{R} : \ |x + y| \leq |x| + |y|$
4. $\forall x, \ y \in \mathbb{R} : \ ||x| - |y|| \leq |x + y|$
5. $\forall \varepsilon > 0, \ \forall x \in \mathbb{R} : \ |x - a| < \varepsilon \Leftrightarrow a - \varepsilon \leq x \leq \varepsilon + a$

1.8 The integer part

Definition 1.8.1. *Let $x \in \mathbb{R}$, there exists a relative integer denoted $E(x)$ such that: $E(x) \leq x \leq E(x) + 1$.*
Is the largest integer less than or equal to x.

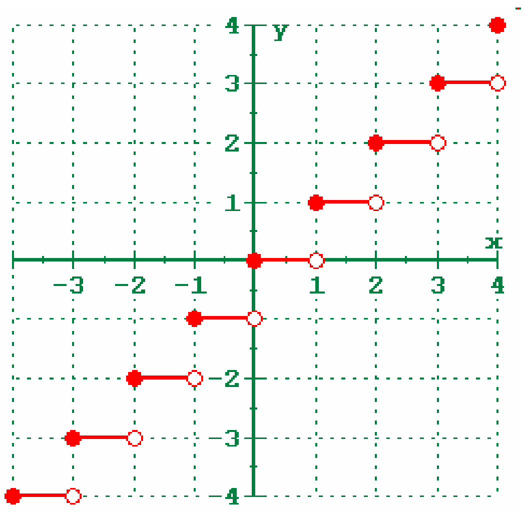

Figure 1.2: graphical presentation of $y = E(x)$

Example 1.8.1.

1) $E(0.3) = 0$, $(0 \leq 0.3 \leq 0 + 1 = 1)$.
2) $E(3.3) = 3$, $(3 \leq 3.3 \leq 3 + 1 = 4)$.
3) $E(-4) = -4$, $E(5) = 5$.
4) $E(-1.5) = -2$, $(-2 \leq -1.5 \leq -2 + 1 = -1)$.

Properties

1. the integer part is an increasing map.
2. $\forall x \in \mathbb{R},\ x = E(x) \Leftrightarrow x \in \mathbb{Z}$.
3. $\forall (x, n) \in (\mathbb{R} \times \mathbb{Z}) : E(x + n) = E(x) + n$.

1.9 Order in $\mathbb{R}$

In order to distinguish the real numbers from all other ordered fields, we will need one additional axiom, to which we now turn. This axiom uses the concepts of upper bounds and least upper bounds, while we are at it, we will also define the related concepts of lower bounds and greatest lower bounds.

Upper bound, Lower bound

Definition 1.9.1. *Let A be a non-empty subset of $\mathbb{R}$, we say that:*

1. *The set A is **bounded above** if there is some $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$. The number M is called an **upper bound** of A.*
2. *The set A is **bounded below** if there is some $m \in \mathbb{R}$ such that $x \geq m$ for all $x \in A$. The number m is called a **lower bound** of A.*
3. *The set A is **bounded** if it is bounded above and bounded below.*
4. *Let $M \in \mathbb{R}$. The number M is a **least upper bound** (also called a **supremum**) of A if M is an upper bound of A, and if $M \leq M'$ for all upper bounds M' of A.*
5. *Let $m \in \mathbb{R}$. The number m is a **greatest lower bound** (also called an **infimum**) of A if m is a lower bound of A, and if $m \geq m'$ for all lower bounds m' of A.*

Maximum, Minimum

Definition 1.9.2. *Let A be a non-empty subset of $\mathbb{R}$, we say that:*

1. *$M \in \mathbb{R}$ is a maximum of A and we denote $\max A$ if $M \in A$ and M is an upper bound of A.*
2. *$m \in \mathbb{R}$ is a minimum of A and we denote $\min A$ if $m \in A$ and m is a lower bound of A.*

Example 1.9.1. 1. *Let $A =]0, 1[$, A is bounded from above by 1 and bounded from below by 0.*

- *The set of upper bounds is $[1, +\infty[$, this one admits the smallest upper bound which is $1 \notin A$. So $\sup(A) = 1$ and $\max(A)$ does not exist.*
- *$]-\infty, 0]$ is the set of lower bounds, this one admits the largest of the lower bounds which is $0 \notin A$. So $\inf(A) = 0$ and $\min(A)$ does not exist.*

2. *Let* $B = \left\{x \in \mathbb{Z} : \ x^2 \leq 49\right\} = \{-7, -6, -5, \cdots, 5, 6, 7\}$.

   - *The set of upper bounds is:* $M = [7, +\infty[$ *and* $7 \in B$. *So* $\sup(B) = \max(B) = 7$.
   - *The set of lower bounds is:* $m =]-\infty, -7]$ *and* $-7 \in B$. *So* $\inf(B) = \min(B) = -7$.

3. $C =]-\infty, 1]$. *So C is bounded above by* $[1, \infty[$*, and not bounded below. Then,* $\max(C) = \sup(C) = 1$ *and* $\inf(C)$, $\min(C)$ *do not exist.*

1.10 The upper and lower bounds characterization

Proposition 1.10.1. *Let A be a non empty subset of* $\mathbb{R}$.

1. *If* $M \in \mathbb{R}$ *is an upper bound of A, then:*

$$M = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A : \ x \leq M, \\ \forall \varepsilon > 0, \ \exists \, x \in A : \ M - \varepsilon < x \leq M. \end{cases}$$

2. *If* $m \in \mathbb{R}$ *is a lower bound of A, then:*

$$m = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A : \ x \geq m, \\ \forall \varepsilon > 0, \ \exists \, x \in A : \ m \leq x < m + \varepsilon. \end{cases}$$

Exercise 1.10.1. *Let* $A = \left\{x_n = \dfrac{1}{2} + \dfrac{n}{2n+1}, \ n \in \mathbb{N}\right\}$.

1. *Show that:* $\forall x_n \in A, \ \dfrac{1}{2} \leq x_n < 1$.
2. *Find* $\sup(A)$, *and* $\inf(A)$.
3. *Show that:* $\sup(A) = 1$.

Solution:

1. *We show that* $\forall x_n \in A, \ \frac{1}{2} \leq x_n < 1$. *We have* $\forall n \in \mathbb{N} : \ x_n = \frac{1}{2} + \frac{n}{2n+1}$. *So*

$$\begin{aligned} \forall n \in \mathbb{N} : \ 0 \leq 2n < 2n+1 \quad &\Rightarrow \quad 0 \leq \tfrac{2n}{2n+1} < 1, \\ &\Rightarrow \quad 0 \leq \tfrac{1}{2} \cdot \tfrac{2n}{2n+1} < \tfrac{1}{2}, \\ &\Rightarrow \quad \tfrac{1}{2} \leq \tfrac{1}{2} + \tfrac{n}{2n+1} < 1. \end{aligned}$$

   So: $\forall n \in \mathbb{N} : \ \dfrac{1}{2} \leq x_n < 1$.

2. *We have* $\frac{1}{2} \leq x_n < 1$, *then A is bounded, i.e:* $\inf(A)$ *and* $\sup(A)$ *are exists.*
   $\frac{1}{2}$ *is a lower bound of A, and* $\frac{1}{2} \in A \Rightarrow \min(A) = \inf(A) = \frac{1}{2}$. *And* 1 *is the smallest upper bound of A, so* $\sup(A) = 1$.

3. *Let's show that:* $\sup(A) = 1$
   We use the characteristic property of the upper bound.

$$\sup(A) = 1 \Longleftrightarrow \begin{cases} 1 \ \textit{is an upper bound of } A, \\ \forall \varepsilon > 0, \ \exists \, x_n \in A \ (n \in \mathbb{N}), \ x_n > 1 - \varepsilon. \end{cases}$$

Assume that: $x_n = \frac{1}{2} + \frac{n}{2n+1} > 1 - \varepsilon$, *and find n as a function of* ε.

$$\begin{aligned} x_n = \tfrac{1}{2} + \tfrac{n}{2n+1} > 1 - \varepsilon \ &\Rightarrow \ -\tfrac{1}{2} + \tfrac{n}{2n+1} > -\varepsilon, \\ &\Rightarrow \ \tfrac{1}{2} - \tfrac{n}{2n+1} < \varepsilon, \\ &\Rightarrow \ \tfrac{1}{2(2n+1)} < \varepsilon, \\ &\Rightarrow \ 2n + 1 > \tfrac{1}{2\varepsilon}, \\ &\Rightarrow \ n > \tfrac{1}{4\varepsilon} - \tfrac{1}{2}. \end{aligned}$$

So $\exists \, n = E(\frac{1}{4\varepsilon} - \frac{1}{2}) + 1$, *thus* $\sup(A) = 1$.

CHAPTER 2

FIELD OF COMPLEX NUMBERS

We know that the square of a real number is always non-negative e.g. $(4)^2 = 16$ and $(-4)^2 = 16$. Therefore, the square root of 16 is (± 4). What about the square root of a negative number? It is clear that a negative number can not have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for the positive square root of -1 i.e., $i = \sqrt{-1}$.

2.1 Definitions and notations

Definition 2.1.1. *A number which can be written in the form $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a* ***complex number****.*

- *If $z = a + ib$ is the complex number, then a and b are called* ***real*** *and* ***imaginary*** *parts, respectively, of the complex number and written as $Re(z) = a$, $Im(z) = b$.*
- *We denote the set of all complex numbers by $\mathbb{C}$.*

- *Order relations "greater than" and "less than" are not defined for complex numbers.*
- *If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if real part is zero, then it is called purely imaginary number, for example,* 2 *is a purely real number because its imaginary part is zero and* 3*i is a purely imaginary number because its real part is zero.*
- *Two complex numbers* $z_1 = a + ib$ *and* $z_2 = c + id$ *are said to be equal if* $a = c$ *and* $b = d$.

2.2 The complex plane

Just as real numbers can be visualized as points on a line, complex numbers can be visualized as points in a plane: plot $z = a + ib$ at the point (a, b).

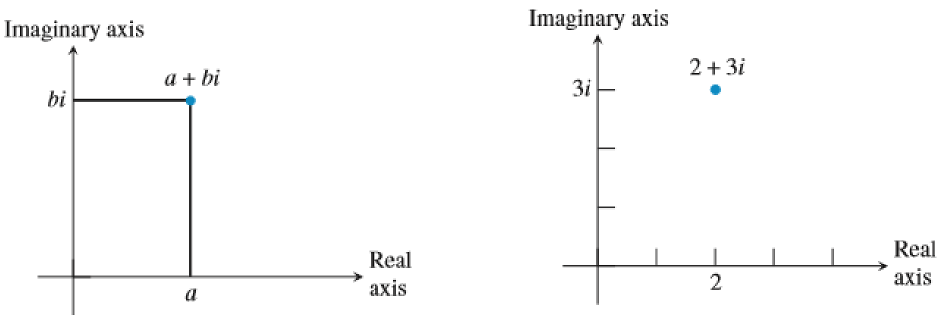

Figure 2.1: Plotting points in the complex plane

2.3 Operations on complex numbers

Addition

- Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1 + z_2 = (a + c) + i(b + d)$.
- Addition of complex numbers is commutative ($z_1 + z_2 = z_2 + z_1$), and associative ($(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$).

- The identity element for addition is 0 ($\forall z = a + ib \in \mathbb{C} :\ \exists\, 0 = 0 + 0i \in \mathbb{C}$ *such that* $z + 0 = 0 + z = z$).
- The additive inverse of z is $-z$ ($\forall z = a + ib \in \mathbb{C} :\ \exists -z = -a - ib \in \mathbb{C}$ *such that* $z + (-z) = (-z) + z = 0$).

Multiplication

- Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1.z_2 = (ac - bd) + i(ad + bc)$.
- Multiplication of complex numbers is commutative ($z_1.z_2 = z_2.z_1$), and associative ($(z_1.z_2).z_3 = z_1.(z_2.z_3)$).
- The identity element for multiplication is 1 ($\forall z \in \mathbb{Z},\ \exists\, 1 = 1 + i0 \in F$ *such that* $z.1 = 1.z = z$).
- The multiplicative inverse of z is $\frac{1}{z}$.
- For complex numbers, multiplication is distributive over addition.

Division

Let $z_1 = z + ib$ and $z_2(\neq 0) = c + id$. Then

$$z_1 \div z_2 = \frac{a + ib}{c + id} = \frac{(ac + bd)}{c^2 + d^2} + i\frac{(bc - ad)}{c^2 + d^2}$$

Conjugate of a complex number

Definition 2.3.1. *In complex numbers, we define something called the conjugate of a complex number which is given by* $\overline{z} = a - ib$. *The conjugate is therefore simply a change in the sign of the imaginary part, i.e.,* ($Re(\overline{z}) = Re(z),$ *and* $Im(\overline{z}) = -Im(z)$).

For example, if $z_1 = 3 + 2i$ then $\overline{z_1} = 3 - 2i$, if $z_2 = -4 - i$ then $\overline{z_2} = -4 + i$, if $z_3 = 5 - 3i$ then $\overline{z_3} = 5 + 3i$.

properties:

1. $\overline{\overline{z}} = z$.

2. $z + \overline{z} = 2Re(z),\ z - \overline{z} = 2iIm(z)$.

3. $z = \overline{z},\ if\ z\ is\ purely\ real.$

4. $z + \overline{z} = 0 \Longleftrightarrow z\ is\ purely\ imaginary.$

5. $z.\overline{z} = \{Re(z)\}^2 + \{Im(z)\}^2$.

6. $(\overline{z_1 + z_2}) = \overline{z_1} + \overline{z_2},\ (\overline{z_1 - z_2}) = \overline{z_1} - \overline{z_2}$.

7. $(\overline{z_1.z_2}) = \overline{z_1}.\overline{z_2},\ (\overline{\frac{z_1}{z_2}}) = \frac{(\overline{z_1})}{(\overline{z_2})},\ (\overline{z_2} \neq 0)$.

Modulus of a complex number

Definition 2.3.2. *Let $z = a + ib$ be a complex number. Then the positive square root of the sum of square of real part and square of imaginary part is called modulus (absolute value) of z and it is denoted by $r = |z|$ i.e., $r = |z| = \sqrt{a^2 + b^2}$.*

Properties:

1. $|z^2| = z \times \overline{z},\ |\overline{z}| = |z|,\ |z_1.z_2| = |z_1|.|z_2|$.

2. $|z| = 0 \Leftrightarrow z = 0$, i.e., $Re(z) = 0$, and $Im(z) = 0$.

3. $|z_1 + z_2| \leq |z_1| + |z_2|$, (Triangle inequality).

4. $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$.

5. $|\frac{1}{z}| = \frac{1}{|z|},\ z \neq 0$.

6. $|Re(z)| \leq |z|$, and $|Im(z)| \leq |z|$.

Proof 1. *(of Triangle inequality)*

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\
&= |z_1|^2 + |z_2|^2 + 2Re(z_1 z_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1 z_2| \\
&\leq (|z_1| + |z_2|)^2
\end{aligned}$$

Argument

Definition 2.3.3. *For any $z \in \mathbb{C}$, a number $\theta \in \mathbb{R}$ such that $z = |z|\,(\cos\theta + i\sin\theta)$ is called an argument of z and denoted by $\theta = arg(z) = \tan^{-1}\frac{b}{a}$.*
the relationship connecting r and θ to a and b is: $a = r\cos\theta$ and $b = r\sin\theta$. i.e.,

$$arg\ (z) = \begin{cases} \cos\theta &= \frac{a}{r}, \\ \sin\theta &= \frac{b}{r}. \end{cases}$$

properties:

1. $arg\ (z_1.z_2) = arg\ (z_1) + arg\ (z_2)$.
2. $arg\ (z^n) = n\ arg\ (z)$.
3. $arg\ (\frac{1}{z} = -\ arg\ (z))$.
4. $arg\ (\overline{z} = -\ arg\ (z))$.
5. $arg\ (\frac{z_1}{z_2}) =\ arg\ (z_1) - arg\ (z_2)$.

2.4 Trigonometric form

Let $z = a + ib$, $r = |z| = \sqrt{a^2 + b^2}$, and $\theta = arg\ (z)$. We have $a = r\ \cos\theta$, $b = r\ \sin\theta$, so:

$$z = a + ib = r\cos\theta + ir\sin\theta = r\,(\cos\theta + i\sin\theta) = r\,e^{i\theta}.$$

This is the trigonometric form of z. This representation is very useful for the multiplication and division of complex numbers:

- $z_1 \times z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1.r_2 e^{i(\theta_1+\theta_2)}$.
- $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

2.5 Inverse Euler formula

Euler's formula gives a complex exponential in terms of sines and cosines. We can turn this around to get the inverse Euler formulas.
Euler's formula says:

$$e^{it} = \cos(t) + i\sin(t) \quad \text{and} \quad e^{-it} = \cos(t) - i\sin(t).$$

By adding and subtracting we get:

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

2.6 Moivre's formula

For positive integers n we have the **Moivre's formula**:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof. This is a simple consequence of Euler's formula:

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

□

Application:

By developing the Moivre's formula using the Newton binomial formula:

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} C_n^k (\cos\theta)^{n-k}(i\sin\theta)^k.$$

Where $C_n^k = \dfrac{n!}{k!(n-k)!}$, $C_n^n = \dfrac{n!}{n!(0)!} = 1$, *and* $C_n^0 = \dfrac{n!}{0!(n)!} = 1$.

We have

$(\cos\theta + i\sin\theta)^n = C_n^0(\cos\theta)^n(i\sin\theta)^0 + C_n^1(\cos\theta)^{n-1}(i\sin\theta)^1 + \cdots + C_n^k(\cos\theta)^{n-k}(i\sin\theta)^k + \cdots + C_n^n(\cos\theta)^0(i\sin\theta)^n$.

So, we get

The real part:

$(\cos n\theta) = (\cos\theta)^n - C_n^2(\cos\theta)^{n-2}(\sin\theta)^2 + C_n^4(\cos\theta)^{n-4}(\sin\theta)^4 + \cdots$

and

The imaginary part:

$(\sin n\theta) = C_n^1(\cos\theta)^{n-1}(\sin\theta)^1 - C_n^3(\cos\theta)^{n-3}(\sin\theta)^3 + \cdots$

Example 2.6.1. *For $n = 3$:*

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^3 &= \textstyle\sum_{k=0}^{3} C_3^k(\cos(\theta))^{3-k}(i\sin(\theta))^k \\ &= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta).\end{aligned}$$

By identifying the real and imaginary parts, we deduce that:

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta), \quad \textit{and} \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

2.7 n-th root of a Complex Number

Definition 2.7.1. *A complex number w is an $n-th$ root of z if:*

$$w^n = z.$$

We use the Moivre's Theorem to develop a general formula for finding the $n-th$ roots of a nonzero complex number. Suppose that $w = \rho(\cos(\theta') + i\sin(\theta'))$ is an $n-th$ root of $z = r(\cos(\theta) + i\sin(\theta))$. Then

$$\begin{cases} w^n = z \\ \rho^n e^{i\,n\,\theta'} = re^{i\theta} \end{cases} \Longrightarrow \begin{cases} \rho^n = r \\ n\theta' = \theta + 2k\pi,\ 0 \le k \le n-1. \end{cases}.$$

So

$$\begin{cases} \rho = \sqrt[n]{r} \\ \theta' = \frac{\theta+2k\pi}{n},\ 0 \le k \le n-1 \end{cases}$$

thus, if $z = r(\cos(\theta) + i\sin(\theta))$*, then the n distinct complex numbers*

$$\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right),\ 0 \le k \le n-1$$

are the $n-th$ *roots of the complex number z.*

Particular case:

If $z = 1$, the $n-th$ roots of 1 are

$$\cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n}),\ 0 \le k \le n-1.$$