Ars le système A : $\begin{cases} -2x + y = -x \\ -4x + 3z = -y \\ y - 2z = -3 \end{cases} \Rightarrow x = y = z.$

$E_{-1} = \langle (1,1,1) \rangle$; $\dim E_{-1} \neq \deg(-1)$, A_{-2} n'est pas diagonalisable.

$\alpha = -3$: (On applique un autre méthode).

On a le théorème du rang : $\dim \mathrm{Ker}(A_{-3} + 2I_2) + \mathrm{rg}(A_{-3} + 2I_2) = \dim \mathbb{R}^3$

$$(A_{-3} + 2I_2) = \begin{pmatrix} -1 & 1 & 0 \\ -6 & 2 & 4 \\ -1 & 1 & 0 \end{pmatrix}, \text{ On a } \mathrm{rg}(A_{-3} + 2I_2) = 2 \Rightarrow \mathrm{Ker}(A_{-3} + 2I_2) = 1$$

Alors ; A_{-3} n'est pas diagonalisable.

On a : $\begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 4 & 1 & -2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$

* les valeurs propre de A_2 sont : $-2, -1, 3$.
* la solution du système : $X(t) = \alpha_1 e^{\lambda_1 t} V_1 + \alpha_2 e^{\lambda_2 t} V_2 + \alpha_3 e^{\lambda_3 t} V_3$.

* $E_{-2} = \{ (x,y,z) \in \mathbb{R}^3 \ / \ (A + 2I_2) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$ soit : $\begin{cases} 4x + y = 0 \\ 4x + 2y - z = 0 \\ 4x + y = 0 \end{cases} \Rightarrow \begin{cases} y = -4x \\ z = -4x \end{cases}$

$E_{-2} = \langle (1,-4,-4) \rangle$.

* $E_{-1} = \{ (x,y,z) \in \mathbb{R}^3 \ | \ (A + I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$ soit : $\begin{cases} 3x + y = 0 \\ 4x + y - z = 0 \\ 4x + y - z = 0 \end{cases} \Rightarrow \begin{cases} y = -3x \\ z = x \end{cases}$

$E_{-1} = \langle (1,-3,1) \rangle$

* $E_3 = \{ (x,y,z) \in \mathbb{R}^3 \ / \ (A - 3I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$ soit : $\begin{cases} -x + y = 0 \\ 4x - 3y - z = 0 \\ 4x + y - 5z = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ z = x \end{cases}$

$E_3 = \langle (1,1,1) \rangle$.

Alors : $X(t) = \alpha_1 e^{-t}\begin{pmatrix}1\\-3\\1\end{pmatrix} + \alpha_2 e^{-2t}\begin{pmatrix}1\\-4\\-4\end{pmatrix} + \alpha_3 e^{3t}\begin{pmatrix}1\\1\\1\end{pmatrix}$

$$= \begin{pmatrix}\alpha_1 e^{-t} + \alpha_2 e^{-2t} + \alpha_3 e^{3t}\\ -3\alpha_1 e^{-t} - 4\alpha_2 e^{-2t} + \alpha_3 e^{3t}\\ \alpha_1 e^{-t} - 4\alpha_2 e^{-2t} + \alpha_3 e^{3t}\end{pmatrix} = \begin{pmatrix}0\\-5\\-1\end{pmatrix} \Rightarrow \alpha_1 = 1,\ \alpha_2 = \frac{1}{5},\ \alpha_3 = -\frac{6}{5}$$

La solution est :
$$\begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} = \begin{pmatrix}e^{-t} + \frac{1}{5}e^{-2t} - \frac{6}{5}e^{3t}\\ -3e^{-t} + \frac{4}{5}e^{-2t} - \frac{6}{5}e^{3t}\\ e^{-t} - \frac{4}{5}e^{-2t} - \frac{6}{5}e^{3t}\end{pmatrix}$$

Exercice 04 : Soit $A = \begin{pmatrix}-4 & -6 & 0\\ 3 & 5 & 0\\ 3 & 6 & 5\end{pmatrix}$

1) Diagonaliser A : $P_A(\lambda) = \begin{vmatrix}-4-\lambda & -6 & 0\\ 3 & 5-\lambda & 0\\ 3 & 6 & 5-\lambda\end{vmatrix} = (5-\lambda)\begin{vmatrix}-4-\lambda & -6\\ 3 & 5-\lambda\end{vmatrix}$

$$= (5-\lambda)\left((-4-\lambda)(5-\lambda)+18\right)$$
$$= (5-\lambda)(\lambda+1)(\lambda-2).$$

3 v.p distinctes alors A est diagonalisable.

* $E_{-1} = \left\{(x,y,z)\in\mathbb{R}^3 \,/\, (A+I_3)\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ soit $\begin{cases}-3x-6y=0\\3x+6y=0\\3x+6y+6z=0\end{cases} \Rightarrow \begin{cases}y=-\frac{1}{2}x\\z=0\end{cases}$

$E_{-1} = \langle(2,-1,0)\rangle.$

* $E_2 = \left\{(x,y,z)\in\mathbb{R}^3 \,/\, (A-2I_3)\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ soit $\begin{cases}-6x-6y=0\\3x+3y=0\\3x+6y+3z=0\end{cases} \Rightarrow \begin{cases}y=-x\\z=x\end{cases}$

$E_2 = \langle(1,-1,1)\rangle.$

* $E_5 = \left\{(x,y,z)\in\mathbb{R}^3 \,/\, (A-5I_3)\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$ soit $\begin{cases}-9x-6y=0\\3x=0\\3x+6y=0\end{cases} \Rightarrow x=y=0$

$E_5 = \langle(0,0,1)\rangle.$

Donc : $A^n = PD^nP^{-1}$; $P^{-1} = \begin{pmatrix}1 & 1 & 0\\ -1 & -2 & 0\\ 1 & 2 & 1\end{pmatrix}.$

$$A^n = \begin{pmatrix}2 & 1 & 0\\ -1 & -1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}-1^n & 0 & 0\\ 0 & 2^n & 0\\ 0 & 0 & 1^n\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ -1 & -2 & 0\\ 1 & 2 & 1\end{pmatrix} = \begin{pmatrix}2(-1)^n + (-2)^n & 2(-1)^n - 2^{n+1} & 0\\ -(-1)^n + 2^n & -(-1)^n + 2^{n+1} & 0\\ -2^n + 1^n & -2^{n+1} + 1^n & 1^n\end{pmatrix}$$

3)

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} -4 & -6 & 0 \\ 3 & 5 & 0 \\ 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$$

$$X_n = A^n X_0$$

$$X_n = \begin{pmatrix} (2(-1)^n - 2^n)u_0 + (2(-1)^n - 2^{n+1})v_0 \\ (-(-1)^n + 2^n)u_0 + (-(-1)^n + 2^{n+1})v_0 \\ (-2^n + 1^n)u_0 + (-2^{n+1} + 1^n)v_0 + w_0 \end{pmatrix}$$

Fin