④ $f_4(t) = t\,e^{-t}\cos t$

Soit : $f(t) = \cos t \Rightarrow F(P) = \dfrac{P}{P^2+1}$

Soit $g(t) = t\cos t$

Donc $G(P) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t\cos t\}$

$$\stackrel{\text{Proposition 1}}{=} (-1)\cdot F'(P)$$

$$= (-1)\left(\frac{P}{P^2+1}\right)'$$

$$= (-1)\cdot\frac{P^2+1-P(2P)}{(P^2+1)^2}$$

$$G(P) = \frac{P^2-1}{(P^2+1)^2}.$$

D'où $\mathcal{L}(f_4(t)) = \mathcal{L}(t\,e^{-t}\cos t)$

$$= \mathcal{L}(e^{-t}g(t))$$

$$\stackrel{\text{Propriété 03}}{=} G(P+1) \qquad \boxed{\alpha = 1}$$

$$= \frac{(P+1)^2-1}{((P+1)^2+1)^2}$$

$$= \frac{P^2+2P}{(P^2+2P+2)^2}$$

Solutions d'exercices (Série 3)

Exercice 01 :

① $f_1(t) = e^{at}\cos wt$

Nous avons : $\mathcal{L}\{e^{-\alpha x}f(x)\} = F(P+\alpha)$ (propriété 03)

et $\mathcal{L}\{\cos wt\} = \dfrac{P}{P^2+w^2} = F(P)$

donc $\mathcal{L}\{e^{at}\cos wt\} \stackrel{\alpha=-a}{=} F(P-a)$

$$= \frac{P-a}{(P-a)^2+w^2}$$

② $f_2(t) = e^{at}\sin wt$ للطالب

③ $f_3(t) = \cos^2 t$

Nous avons : $\cos^2 t = \dfrac{1+\cos 2t}{2}$

donc : $\mathcal{L}\{f_3(t)\} = \mathcal{L}\{\cos^2 t\} = \mathcal{L}\left\{\dfrac{1+\cos 2t}{2}\right\}$

$$= \frac{1}{2}\cdot\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{2}\cdot\frac{1}{P} + \frac{1}{2}\cdot\frac{P}{P^2+4}$$

$$= \frac{1}{2}\left(\frac{1}{P} + \frac{P}{P^2+4}\right).$$

② La transformée de Laplace de f_k:

$$f_k(t) = \frac{t^k}{k!} e^{at} \quad , k \in \mathbb{N}$$

Pour $k=0$, $f_0(t) = e^{at} \longrightarrow \mathcal{L}\{f_0(t)\} = \frac{1}{p-a}$

Pour $k=1$: $f_1(t) = t e^{at} = f(t)$

$$\mathcal{L}\{f_1(t)\} = \frac{1}{(p-a)^2}$$

donc ⊛ est vrai pour $k=0$ et $k=1$.

supposons que ⊛ et vrai pour k et on montre qu'elle vraie pour $k+1$

$$f_{k+1}(t) = \frac{t^{k+1}}{(k+1)!} e^{at} \Rightarrow$$

$$\mathcal{L}\{f_{k+1}(t)\} = \frac{1}{(k+1)!} \int_0^\infty t^{k+1} e^{at} e^{-pt} dt$$

$$= \frac{1}{(k+1)!} \int_0^\infty t^{k+1} e^{-(p-a)t} dt$$

Re $p > a$

$$= \frac{1}{(k+1)!} \left[\cancel{t^{k+1} \cdot \frac{-1}{p-a} e^{-(p-a)t} \Big|_0^\infty} + \frac{1}{p-a} \int_0^\infty (k+1) t^k e^{-(p-a)t} dt \right]$$

$$= \frac{1}{(k+1)!} \cdot (k+1) \cdot \frac{1}{p-a} \int_0^\infty t^k e^{-(p-a)t} dt$$

$$= \frac{1}{p-a} \int_0^\infty \frac{1}{k!} t^k e^{-(p-a)t} dt$$

Exo 2:

① La transformée de Laplace de f:

1ère Méthode:

$$f(t) = t e^{at}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-pt} dt$$

$$= \int_0^\infty t e^{(a-p)t} dt$$

$$= \int_0^\infty t e^{-(p-a)t} dt$$

Re $p > a$

$$= \cancel{t \cdot \frac{-1}{p-a} e^{-(p-a)t} \Big|_0^\infty} + \frac{1}{p-a} \int_0^\infty e^{-(p-a)t} dt$$

$$= \frac{1}{p-a} \cdot \left(\frac{-1}{p-a} e^{-(p-a)t} \Big|_0^\infty \right)$$

Re $p > a$

$$= \frac{-1}{(p-a)^2} \left(\cancel{e^{-(p-a)\infty}} - e^0 \right)$$

$$\mathcal{L}(te^{at}) = \frac{1}{(p-a)^2}$$

2ème Méthode: soit $g(t) = t \Rightarrow G(p) = \frac{1}{p^2}$

donc $\mathcal{L}(te^{at}) = \mathcal{L}(e^{at} g(t)) = G(p-a)$

$$= \left(\frac{1}{p-a}\right)^2.$$

$$B = (p+2)^2 F(p)\Big|_{p=-2} = 3p\Big|_{p=-2} = -6.$$

$$A = \frac{d}{dp}\left[(p+2)^2 F(p)\right]_{p=-2} = \frac{d}{dp}\left(3p\Big|_{p=-2}\right) = 3$$

donc $F(p) = \frac{3}{p+2} - \frac{6}{(p+2)^2}$.

$$f(t) = \mathcal{L}^{-1}(F(p)) = 3.\mathcal{L}^{-1}\left(\frac{1}{p+2}\right) - 6.\mathcal{L}^{-1}\left(\frac{1}{(p+2)^2}\right)$$

$$= 3e^{-2t} - 6\mathcal{L}^{-1}\left(\frac{1}{(p+2)^2}\right)$$

$k=1, a=-2$ $\circledast$ $= 3e^{-2t} - 6te^{-2t}$.

ⓑ $F(p) = \frac{e^{-3p}}{p^2(p-1)} = e^{-3p}\left(\underbrace{\frac{Ap+B}{p^2}}\; + \frac{C}{p-1}\right)$ $\left(\frac{A}{p} + \frac{B}{p^2} + \frac{C}{p-1}\right)$

$$= e^{-3p}\left[\frac{-p-1}{p^2} + \frac{1}{p-1}\right]$$

$$= e^{-3p}\left[\frac{1}{p-1} - \frac{1}{p} - \frac{1}{p^2}\right]$$

donc $f(t) = \mathcal{L}^{-1}\left(\frac{e^{-3p}}{p^2(p-1)}\right)$

$$= \mathcal{L}^{-1}\left(\frac{e^{-3p}}{p-1}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3p}}{p}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3p}}{p^2}\right)$$

Nous avons : $L\{f(x-c)\} = e^{-cp}F(p)$

donc. $\mathcal{L}^{-1}\left(e^{-cp}F(p)\right) = f(x-c)$.

$$\overset{\circledast}{=} \frac{1}{(p-a)} \cdot \frac{1}{(p-a)^{k+1}} = \frac{1}{(p-a)^{k+2}}$$

D'où $\mathcal{L}(f_k)(p) = \frac{1}{(p-a)^{k+1}}$.

Exercice 03 :

La transformée inverse de Laplace.

ⓐ $F(p) = \frac{2p+1}{p^2+5p+6} = \frac{A}{p+3} + \frac{B}{p+2}$

$$A = (p+3)F(p)\Big|_{p=-3} = \frac{2p+1}{p+2}\Big|_{p=-3} = \frac{-5}{-1} = 5$$

$$B = (p+2)F(p)\Big|_{p=-2} = \frac{2p+1}{p+3}\Big|_{p=-2} = \frac{-3}{1} = -3$$

donc $F(p) = \frac{5}{p+3} - \frac{3}{p+2}$

$$f(t) = \mathcal{L}^{-1}(F(p)) = \mathcal{L}^{-1}\left(\frac{5}{p+3}\right) - \mathcal{L}^{-1}\left(\frac{3}{p+2}\right)$$

$$f(t) = 5e^{-3t} - 3e^{-2t}$$

ⓒ $F(p) = \frac{3p}{p^2+4p+4} = \frac{3p}{(p+2)^2}$

$$= \frac{A}{p+2} + \frac{B}{(p+2)^2}$$

p.

* Si $F_1(P) = \frac{1}{P-1} \longrightarrow f_1(t) = e^t$

donc $\mathcal{L}^{-1}\left(e^{-3P} \cdot \frac{1}{P-1}\right) = f_1(t-3)$
$= e^{t-3}$

* Si $F_2(P) = \frac{1}{P} \longrightarrow f_2(t) = 1$

donc $\mathcal{L}^{-1}\left(e^{-3P} \cdot \frac{1}{P}\right) = f_2(t-3) = 1.$

* Si $F_3(P) = \frac{1}{P^2} \Longrightarrow f_3(t) = t$

donc $\mathcal{L}^{-1}\left(e^{-3P} \cdot \frac{1}{P^2}\right) = f_3(t-3) = t-3.$

D'où $f(t) = e^{t-3} - 1 - (t-3)$

$f(t) = e^{t-3} - t + 2$, $(t-3) \in [0, +\infty[$

Remarque: Les fonctions f_i, sont des fonctions causales i.e $f_i(t) = H(t-3) f_i(t)$

$$= \begin{cases} f_i(t) & \text{si } t-3 \geq 0 \\ 0 & \text{si } t-3 < 0 \end{cases}$$

donc $f(t) = H(t-3) \cdot \left[e^{t-3} - t + 2\right]$

Exercice 04:

① $y''(t) - y(t) = 3e^{-2t} + t + 1$, $y(0) = 0 = y'(0)$.

$\Longrightarrow \mathcal{L}\left(y''(t) - y(t)\right) = \mathcal{L}\left(3e^{-2t} + t + 1\right)$

$\Longrightarrow \mathcal{L}(y'')(P) - \mathcal{L}(y)(P) = 3\mathcal{L}(e^{-2t})(P) + \mathcal{L}(t)(P) + \mathcal{L}(1)(P)$

$\Longrightarrow P^2 Y(P) - y'(0) - P y(0) - Y(P) = \frac{3}{P+2} + \frac{1}{P^2} + \frac{1}{P}$

$\Longrightarrow (P^2 - 1) Y(P) = \frac{3}{P+2} + \frac{1}{P^2} + \frac{1}{P} = \frac{3P^2 + P + 2 + P(P+2)}{P^2(P+2)}$

$\Longrightarrow Y(P) = \frac{4P^2 + 3P + 2}{P^2(P+2)(P^2-1)}$

$$= \frac{A_1}{P} + \frac{A_2}{P^2} + \frac{A_3}{P+2} + \frac{A_4}{P-1} + \frac{A_5}{P+1}$$

$A_1 = \frac{d}{dP}\left[P^2 Y(P)\right]_{P=0} = -1$

$A_2 = P^2 Y(P)\Big|_{P=0} = \frac{4P^2 + 3P + 2}{(P+2)(P^2-1)}\Big|_{P=0} = -1$

$A_3 = (P+2) Y(P)\Big|_{P=-2} = \frac{16 - 6 + 2}{4 \cdot 3} = 1$

$A_4 = (P-1) \cdot Y(P)\Big|_{P=1} = \frac{4+3+2}{1 \cdot 3 \cdot 2} = \frac{9}{6} = \frac{3}{2}$

$A_5 = (P+1) Y(P)\Big|_{P=-1} = \frac{4 - 3 + 2}{1 \cdot 1 \cdot (-2)} = -\frac{3}{2}$

③ $y''(t) + 3y = \sin(t)$, $y(0)=1$, $y'(0)=2$.

$\Rightarrow p^2 Y(p) - p y(0) - y'(0) + 3Y(p) = \frac{1}{p^2+1}$

$\Rightarrow Y(p)(p^2+3) = p+2 + \frac{1}{p^2+1}$

$\Rightarrow Y(p) = \frac{p+2}{p^2+3} + \frac{1}{(p^2+3)(p^2+1)}$

$= \frac{p+2}{p^2+3} + \frac{-\frac{1}{2}}{p^2+3} + \frac{\frac{1}{2}}{p^2+1} = \frac{p+\frac{3}{2}}{p^2+3} + \frac{\frac{1}{2}}{p^2+1}$

$= \frac{p}{p^2+3} + \frac{\frac{\sqrt{3}}{2}\cdot\sqrt{3}}{p^2+(\sqrt{3})^2} + \frac{1}{2}\cdot\frac{1}{p^2+1}$

D'où $y(t) = \cos\sqrt{3}t + \frac{\sqrt{3}}{2}\cdot\sin\sqrt{3}t + \frac{1}{2}\sin(t)$

④ $y'''(t) + 5y''(t) + 6y'(t) = 0$, $y(0)=3$, $y'(0)=-2$, $y''(0)=7$.

$\Rightarrow p^3 Y(p) - p^2 y(0) - p y'(0) - y''(0) + 5\left(p^2 Y(p) - p y(0) - y'(0)\right)$

$+ 6\left(p Y(p) - y(0)\right) = 0$

$\Rightarrow (p^3 + 5p^2 + 6p) Y(p) - 3p^2 + 2p - 7 - 15p + 10 - 18 = 0$

$\Rightarrow Y(p) = \frac{3p^2 + 13p + 15}{p(p^2+5p+6)} = \frac{3p^2+13p+15}{p(p+2)(p+3)}$

$= \frac{5}{2}\cdot\frac{1}{p} - \frac{1}{2}\cdot\frac{1}{p+2} + \frac{1}{p+3}$

D'où $y(t) = \frac{5}{2} - \frac{1}{2}e^{-2t} + e^{-3t}$

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donc $Y(p) = \frac{-1}{p} - \frac{1}{p^2} + \frac{1}{p+2} + \frac{3}{2}\cdot\frac{1}{p-1} - \frac{3}{2}\,\frac{1}{p+1}$

d'où $y(t) = \mathcal{L}^{-1}\left(-\frac{1}{p} - \frac{1}{p^2} + \frac{1}{p+2} + \frac{3}{2}\cdot\frac{1}{p-1} - \frac{3}{2}\cdot\frac{1}{p+1}\right)$

$y(t) = -1 - t + e^{-2t} + \frac{3}{2}e^{t} - \frac{3}{2}e^{-t}$.

② $y''(t) + 2y'(t) + 5y(t) = 0$, $y(0)=2$, $y'(0)=-4$.

$\Rightarrow \mathcal{L}(y''(t)) + 2\mathcal{L}(y'(t)) + 5\mathcal{L}(y(t)) = \mathcal{L}(0)$

$\Rightarrow p^2 Y(p) - p y(0) - y'(0) + 2\left(p Y(p) - y(0)\right) + 5Y(p) = 0$

$\Rightarrow (p^2 + 2p + 5) Y(p) - 2p + 4 - 4 = 0$

$\Rightarrow Y(p) = \frac{2p}{p^2+2p+5}$

$= \frac{2p}{p^2+2p+1+4} = \frac{2p+2-2}{(p+1)^2+2^2}$

$= \frac{2(p+1)}{(p+1)^2+2^2} - \frac{2}{(p+1)^2+2^2}$

donc $y(t) = 2.\,\mathcal{L}^{-1}\left(\frac{p+1}{(p+1)^2+2^2}\right) - \mathcal{L}^{-1}\left(\frac{2}{(p+1)^2+2^2}\right)$

$y(t) = 2e^{-t}\cos 2t - e^{-t}\sin 2t$

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(d) Nous avons : (Exo3)

$$F(P) = \frac{1}{P(P^2+w^2)}$$

Comme $\mathcal{L}^{-1}\left(\frac{1}{p^2+w^2}\right) = \frac{\sin wt}{w}$

alors : $\mathcal{L}^{-1}\left(\frac{1}{P(P^2+w^2)}\right) = \frac{1}{w}\int_0^t \sin wx \, dx$

$\left[\text{Car } \mathcal{L}\left(\int_0^x f(t)\,dt\right) = \frac{F(P)}{P}\right]$

donc

$$f(t) = \frac{1}{w}\int_0^t \sin wx \, dx$$

$$= \frac{1}{w}\cdot\frac{1}{w}\left(-\cos wx \Big|_0^t\right)$$

$$\boxed{f(t) = \frac{1}{w^2}\left(1-\cos t\right)}$$

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