Série TD N° 02 : Les Variable aléatoire

1. Varibles aléatoire discrètes :

Exercice 01 :

$\Omega : \{(a,b,c),\ a,b \text{ et } c \in [1,2,3,4,5,6]\}$

$\Omega = \{1,2,3,4,5,6\}^3$

$\Omega = \{(1,1,1),(1,1,2),(1,1,3),\dots,(6,6,6)\}$

$\text{card } \Omega = 6^3 = 216$.

2/ La loi de X :

$X = [1,2,3]$

$P(X=1) = \frac{6}{6^3} = \boxed{\frac{1}{36}}$

$P(X=3) = \frac{A_6^3}{6^3} = \frac{\frac{6!}{(6-3)!}}{6^3} = \frac{\frac{6\times5\times4\times\cancel{3!}}{\cancel{3!}}}{6^3}$

$P(X=3) = \frac{20}{36}$

$P(X=2) = \frac{3\times6\times5}{6^3} = \boxed{\frac{15}{36}}$ ....(1)

$P(X=2) = P(X=\Omega) - [P(X=3) + P(X=1)]$

$= \frac{36 - [20+1]}{36}$

$P(X=2) = \frac{15}{36}$ .... (2)

On vérifie que $\sum_{i=1}^{3} p_i = 1$

$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} P(X=x_i)$

$= P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{36} + \frac{15}{36} + \frac{20}{36} = \frac{36}{36}$

$\sum_{i=1}^{3} p_i = 1$

Exercice 02 :

1) La fonction de répartition de :

On a $X = \{0,2,4\}$

- Si $x < 0$ $F(x) = 0$
- Si $0 \leq x < 2$ $F(x) = P(X \leq x)$

$= P(x=0) = \boxed{\frac{21}{32}}$

- Si $2 \leq x < 4$ $F(x) = P(X \leq x)$

$= P(x=0) + P(x=2)$

$= \frac{21}{32} + \frac{6}{32}$

$= \boxed{\frac{27}{32}}$

- Si $x \geq 4$. $F(x) = P(X \leq x)$

$\frac{A_6^2}{216} = \frac{\frac{6!}{4!}}{216} = \frac{6\times5}{216} = \frac{30}{216}$

$= P(x=0) + P(x=2) + P(x=4)$

$= \frac{21+6+5}{32} = 1$

Exercice 4

$C = \frac{1}{4}$

$$f(x) = \begin{cases} \frac{1}{4}x & \text{si } 0 \leq x < 2 \\ \frac{1}{4}(4-x) = 1 - \frac{x}{4} & \text{si } 2 \leq x < 4 \\ 0 & \text{ailleurs} \end{cases}$$

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

calculer la moyenne $E(X)$

$$E(X) = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{2} x f(x)\,dx + \int_{2}^{4} x f(x)\,dx + \int_{4}^{+\infty} x f(x)\,d(x)$$

$$= \int_0^2 x\left(\frac{1}{4}x\right)dx + \int_2^4 x f(x)\,dx$$

$$\int_0^2 \frac{1}{4}x^2\,dx + \int_2^4 x - \frac{x^2}{4}\,dx$$

$$\left.\frac{x^3}{12}\right]_0^2 + \left.\frac{x^2}{2} - \frac{x^3}{12}\right]_2^4$$

$$\frac{8}{12} - 0 + \left(\frac{16}{2} - \frac{64}{12}\right) - \left(\frac{4}{2} - \frac{8}{12}\right)$$

2

$$\boxed{E(X) = 2}$$

$E(X)^2 = 4$

$$\boxed{E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx}$$

$$E(X^2) = \int_{-\infty}^{0} x^2 f(x)\,dx + \int_0^2 x^2 f(x)\,dx + \int_2^4 x^2 f(x)\,dx + \int_4^{+\infty} x^2 f(x)\,dx$$

$$E(X^2) = \int_0^2 x\left(\frac{1}{4}x\right)dx + \int_2^4 x(1-\frac{x}{4})$$

$$= \int_0^2 \frac{x^3}{4}\,dx + \int_2^4 \left(x^2 - \frac{x^3}{4}\right)dx$$

$$= \left.\frac{x^4}{16}\right]_0^2 + \left.\frac{x^3}{3} - \frac{x^4}{16}\right]_2^4$$

$$= 1 + \left(\frac{64}{3} - \frac{256}{16}\right) - \left(\frac{8}{3} - 1\right)$$

$$= 1 + \frac{256}{48} - \frac{5}{3} = 1 + \frac{528}{144}$$

$$= \frac{672}{144} = \frac{14}{3}$$

$$\boxed{E(X^2) = \frac{14}{3}}$$

3/ $Var(X) = E(X^2) - [E(X)]^2$

$Var(X) = \frac{14}{3} - 4$

$$\boxed{Var(X) = \frac{2}{3}}$$

4/ $\sigma(X) = \sqrt{Var(X)}$

$$\sigma(X) = \sqrt{\frac{2}{3}}$$

5/ La fonction de répartition

discrète

$F(x) = P(X \leq x) = \sum p_i$

Si $x<0 \Rightarrow F(x)=\int_{-\infty}^{x} f(t)\,dt = 0$

Si $0 \le x < 2 \Rightarrow$

$F(x)=\int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$

$=\int_{0}^{x} f(t)\,dt = \int_{0}^{x} \frac{1}{4} t\,dt$

$=\frac{1}{8} t^2 \Big|_{0}^{x}$

$=\frac{1}{8} x^2$

Si $2 \le x < 4 \Rightarrow F(x)=\int_{-\infty}^{0} f(t)\,dt +$

$\int_{0}^{2} f(t)\,dt + \int_{2}^{x} f(t)\,dt$

$\int_{0}^{2} \frac{1}{4} t\,dt + \int_{2}^{x} \frac{1}{4}(4-t)\,dt$

$\frac{1}{8} t^2 \Big|_{0}^{2} + \left(t - \frac{t^2}{8}\right)\Big|_{2}^{x}$

$=\frac{1}{2} + \left(x - \frac{x^2}{8}\right) - \left(2 - \frac{1}{2}\right)$

$\boxed{x - \frac{x^2}{8} - 1}$

Si $x \ge 4 \Rightarrow F(x)=\int_{-\infty}^{0} f(t)\,dt$

$\int_{0}^{2} f(t)\,dt + \int_{2}^{4} f(t)\,dt + \int_{4}^{x} f(t)\,dt$

$\frac{1}{8} t^2 \Big|_{0}^{2} + t - \frac{t^2}{8} - \frac{1}{2}\Big|_{2}^{4}$

$\frac{1}{2} + 4 - 2 - 1 - 2 + \frac{1}{2} + 1$

$\boxed{= 1}$

$$F(x)=\begin{cases} 0 & \text{si } x<0 \\ \frac{1}{8}x^2 & \text{si } 0 \le x < 2 \\ x - \frac{x^2}{8} - 1 & \text{si } 2 \le x < 4 \\ 1 & \text{si } x \ge 4 \end{cases}$$

Exercice 05 :

1) on détermine c :

$$f(x)\begin{cases} \frac{3}{2}x^2 & \text{si } -c < x < c \\ 0 & \text{si non;} \end{cases}$$

f densité $\Leftrightarrow \begin{cases} f \ge 0; \\ \int_{-\infty}^{+\infty} f(x)\,dx = 1 \end{cases}$

$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{-c} f(x)\,dx + \int_{-c}^{c} \frac{3}{2} x^2\,dx +$

$\int_{c}^{+\infty} f(x)\,dx = 1$

$= \frac{3}{2} \int_{-c}^{c} x^2\,dx = \frac{3}{2} \frac{x^3}{3} \Big|_{-c}^{c} = 1$

$= \frac{c^3}{2} + \frac{c^3}{2} = 1$

$c^3 = 1$

$\boxed{c = 1}$