Exo3 | Soit $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 4 & 1 & -2 \\ 2 & 1 & 2 & -1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$

1) **Calculer $Sp(A)$**: $P_A(\lambda) = (1-\lambda) \begin{vmatrix} 4-\lambda & 1 & -2 \\ 1 & 2-\lambda & -1 \\ 2 & 1 & -\lambda \end{vmatrix}$ $C_1 \leftarrow C_1 + C_3$

$P_A(\lambda) = (1-\lambda) \begin{vmatrix} 2-\lambda & 1 & -2 \\ 0 & 2-\lambda & -1 \\ 2-\lambda & 1 & -\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) \begin{vmatrix} 1 & 1 & -2 \\ 0 & 2-\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix}$ $L_1 \leftarrow L_3 - L_1$

$= (1-\lambda)(2-\lambda) \begin{vmatrix} 0 & 0 & 2-\lambda \\ 0 & 2-\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)(2-\lambda)(\lambda-2)$

$P_A(\lambda) = (\lambda-1)(2-\lambda)^3$; $Sp(A) = \{ \overset{\text{simple}}{1}, \overset{\text{triple}}{2} \}$.

2) **Jordanisation**: * $E_2 = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, (A-2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

Alors: $\begin{cases} -x = 0 \\ -x + 2y + z - 2t = 0 \\ 2x + y - t = 0 \\ x + 2y + z - 2t = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = t \\ z = 0 \end{cases}$ $E_2 = \langle (0,1,0,1) \rangle$.

* $(A-2I)\,\omega = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} -x = 0 \\ -x + 2y + z - 2t = 1 \\ 2x + y - t = 0 \\ x + 2y + z - 2t = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = t \\ z = 1 \end{cases}$; $\omega = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

* $(A-2I)\,\omega_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} -x = 0 \\ -x + 2y + z - 2t = 1 \\ 2x + y - t = 1 \\ x + 2y + z - 2t = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1+t \\ z = -1 \end{cases}$; $\omega_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

* $E_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, (A-I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$; $\begin{cases} -x + 3y + z - 2t = 0 \Rightarrow z = 2t - 2x \\ 2x + y + z - t = 0 \\ x + 2y + z - t = 0 \end{cases} \Big\} x = y$

$E_1 = \langle (1,1,0,1) \rangle$

Donc: $A = PJP^{-1}$ **avec**: $J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ et $P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{pmatrix}$

Exercice (04) Soit $A_k = \begin{pmatrix} 0 & k & -k \\ -1 & k+1 & -k \\ 0 & 0 & 1 \end{pmatrix}$

1) $P_{A_k}(\lambda)$?

$$P_{A_k}(\lambda) = \begin{vmatrix} -\lambda & k & -k \\ -1 & k+1-\lambda & -k \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & k \\ -1 & k+1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\lambda^2 - (k+1)\lambda + k\right)$$

$$= (1-\lambda)(1-\lambda)(k-\lambda)$$

$$= (1-\lambda)^2 (k-\lambda)$$

2) Si $k=1$: $P_{A_1}(\lambda) = (1-\lambda)^3$. Alg:

$$A_1 = PDP^{-1} = P I_3 P^{-1} = I_3 \neq A_1$$

Donc : A_1 n'est pas diagonalisable

3) Trigonalisation A_1 : On calcule les sous-espaces propres de A_1 :

$$E_1 = \left\{ (x,y,z) \in \mathbb{R}^3 \,/\, (A-I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Soit : $\begin{cases} -x + y - z = 0 \\ -x + y - z = 0 \end{cases} \Rightarrow x = y - z$

$$E_1 = \{ (y-z, y, z) \,,\, y, \in \mathbb{R} \} = \langle \overbrace{(1,1,0)}^{v_1}, \overbrace{(-1,0,1)}^{v_2} \rangle$$

complétons nous la base de E_1 par :

$(A-I)\omega = v_1 \Leftrightarrow \begin{cases} -x + y - z = 1 \\ -x + y - z = 1 \end{cases} \Rightarrow x = y - z + 1$

$$E_1 = \{ (y-z+1, y, z) \,;\, y, z \in \mathbb{R} \} = \langle (1,1,0), (-1,0,1), (1,0,0) \rangle$$

Donc : $P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ et $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$A_1 = P J P^{-1}$$