

Propriété:

$$F(F(f(x))) = \hat{\hat{f}}(x) = 2\pi f(-x)$$

Preuve:

Nous avons: $\hat{f}(w) = \int_{-\infty}^{+\infty} f(x) e^{-iwx} dx$

et $F^{-1}(\hat{f}(w)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(w) e^{iwx} dw$

\parallel
 $f(x)$

donc $f(-x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(t) e^{-itx} dt$

$$= \frac{1}{2\pi} F(\hat{f}(t))$$

$$= \frac{1}{2\pi} F(F(f(x)))$$

d'où $F(F(f(x))) = 2\pi f(-x)$

Exercice 03:

Nous avons: $\int_{\mathbb{R}} \frac{f(t)}{(x-t)^2 + a^2} dt = \frac{1}{x^2 + b^2} \dots \textcircled{*}$

Soit $g_c(t) = \frac{1}{t^2 + c^2}$ donc $f * g_a = g_b$

En effet:

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$$\begin{aligned}
 f * g_a(x) &= \int_{\mathbb{R}} f(t) g_a(t-x) dt \\
 &= \int_{\mathbb{R}} f(t) \frac{1}{(t-a)^2 + a^2} dt \\
 &\stackrel{(*)}{=} \frac{1}{x^2 + b^2} = g_b(x) \cdot
 \end{aligned}$$

$$f * g_a = g_b \Rightarrow \widehat{f * g_a} = \widehat{g_b}$$

$$\Rightarrow \hat{f} \cdot \hat{g}_a = \hat{g}_b$$

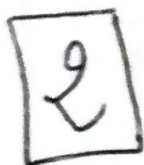
$$\text{d'où } \hat{f} = \frac{\hat{g}_b}{\hat{g}_a}$$

$$\hat{f}(x) = \frac{\hat{g}_b(x)}{\hat{g}_a(x)} \stackrel{(**)}{=} \frac{\frac{\pi}{b} e^{-(b-a)|x|}}{\frac{\pi}{a} e^{-a|x|}} = \frac{a}{b} e^{-(b-a)|x|}$$

$$\text{Donc } \mathcal{F}^{-1}(\hat{f}(x)) = \frac{a}{b} \mathcal{F}^{-1}(e^{-(b-a)|x|})$$

$$\text{f(t)} \stackrel{(***)}{=} \frac{a}{b} \cdot \frac{b-a}{\pi} \cdot \frac{1}{(b-a)^2 + t^2}$$

$$\text{d'où } f(t) = \frac{a \cdot (b-a)}{b \pi [(b-a)^2 + t^2]}.$$



Mentions (**):

Soit $f(x) = e^{-a|x|}$, $a > 0$.

f est une fonction paire alors:

$$F(f(x)) = F(e^{-a|x|}) = 2 \int_0^{+\infty} e^{-ax} \cos wx dx$$

$$= 2 \operatorname{Re} \left(\int_0^{\infty} e^{-ax} e^{iwx} dx \right)$$

$$= 2 \operatorname{Re} \left(\int_0^{\infty} e^{(-a+iw)x} dx \right)$$

$$= 2 \operatorname{Re} \left(\frac{1}{-a+iw} e^{(-a+iw)x} \Big|_0^{\infty} \right)$$

$$= 2 \operatorname{Re} \left(\frac{1}{a-iw} \right) = \frac{2a}{a^2+w^2}$$

Nous avons: $F(F(f(x))) = 2\pi f(-x)$

alors: $F\left(\frac{2a}{a^2+w^2}\right) = 2\pi e^{-a|x|}$

donc $F\left(\frac{1}{a^2+w^2}\right) = \frac{2\pi}{2a} e^{-a|x|}$

d'où $F\left(\frac{1}{a^2+w^2}\right) = \frac{\pi}{a} e^{-a|x|}$

$$F^{-1}\left(e^{-a|x|}\right) = \frac{a}{\pi} \cdot \frac{1}{a^2+w^2} \dots \quad (***)$$

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