Exo 1 :

1/ $f(x,y) = e^{x} \cos y$.

$\frac{\partial f}{\partial x} = e^{x} \cos y$, $\frac{\partial f}{\partial y} = - e^{x} \sin y$.

2/ $f(x,y) = (x^2+y^2) \cos(xy)$.

$\frac{\partial f}{\partial x} = 2x \cos(xy) - y(x^2+y^2) \sin(xy)$.

$\frac{\partial f}{\partial y} = 2y \cos(xy) - x(x^2+y^2) \sin(xy)$.

3) $f(x,y) = \sqrt{1 - x^2y^2}$

$\frac{\partial f}{\partial x} = \frac{-2xy^2}{2\sqrt{1-x^2y^2}} = \frac{-xy^2}{\sqrt{1-x^2y^2}}$

$\frac{\partial f}{\partial y} = \frac{-2yx^2}{2\sqrt{1-x^2y^2}} = \frac{-yx^2}{\sqrt{1-x^2y^2}}$

4/ $f(x,y) = x^2(x+y)$.

$\frac{\partial f}{\partial x} = 2x(x+y) + x^2$.

$\frac{\partial f}{\partial y} = x^2$.

5/ $f(x,y) = e^{x} y$.

$$\frac{\partial f}{\partial x} = e^{x} y \quad , \quad \frac{\partial f}{\partial y} = e^{x}.$$

6) $f(x,y) = h(\underbrace{3x+4}_{u}, \underbrace{-7\sqrt{x}+5}_{v})$. avec : $h(u,v) = u^4 + v^2$

$$\frac{\partial f}{\partial x} = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial x}.$$

$$= 4u^3 \cdot \frac{\partial u}{\partial x} + 2v \cdot \frac{\partial v}{\partial x}$$

$$= 4(3x+4)^3 \cdot 3 + 2(-7\sqrt{x}+5) \cdot \frac{-7}{2\sqrt{x}}.$$

$$= 12(3x+4)^3 - \frac{7}{\sqrt{x}}(-7\sqrt{x}+5)$$

$$\frac{\partial f}{\partial y} = \frac{\partial h}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial y}.$$

$$= 4u^3 \cdot \frac{\partial u}{\partial y} + 2v \cdot \frac{\partial v}{\partial y}.$$

$$= 4(3x+4)^3 \cdot 0 + 2(-7\sqrt{x}+5) \cdot 0 \quad = 0.$$

EXO 2 :

1/ $f(x,y) = \dfrac{x^2 y^2}{x^2+y^2}$

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 0\}$$

$$= \mathbb{R}^2 / \{(0,0)\}.$$

2) $f(x,y) = \dfrac{x^2+y^2}{x+y}$.

$D_f = \{ (x,y) \in \mathbb{R}^2 , x+y \neq 0 \}$

$= \{ (x,y) \in \mathbb{R}^2 , x \neq -y \}$.

3) $f(x,y) = -x^2y + \frac{1}{2}y^2 + y$.

$D_f = \mathbb{R}^2$

4) $f(x,y) = \ln(x^2+y^2) - x^2 - x$.

$D_f = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 > 0 \}$

$= \mathbb{R}^2 \setminus \{(0,0)\}$.

5) $f(x,y) = x e^{xy}$.

$D_f = \mathbb{R}^2$.

6) $f(x,y) = \ln(x + \sqrt{x^2+y^2})$.

$D_f = \{ (x,y) \in \mathbb{R}^2 \mid x > 0 \text{ et } y \neq 0 \}$.

EXO 3: $f(x,y) = \dfrac{x^2}{y-2x^2}$

1) $D_f = \{ (x,y) \in \mathbb{R}^2 \mid y \neq 2x^2 \}$.

2) $\text{grad} f(1,1) = \left(\dfrac{\partial f}{\partial x} , \dfrac{\partial f}{\partial y} \right)(1,1)$

$$\frac{\partial f}{\partial x} = \frac{2x\left(y - 2x^2\right) + 4x\left(x^2\right)}{\left(y - 2x^2\right)^2}$$

$$= \frac{2xy - \cancel{4x^3} + \cancel{4x^3}}{\left(y - 2x^2\right)^2} = \frac{2xy}{\left(y - 2x^2\right)^2}$$

$$\frac{\partial f}{\partial x}(1,1) = \frac{2(1)(1)}{\left(1 - 2(1)^2\right)^2} = \frac{2}{(-1)^2} = 2.$$

$$\frac{\partial f}{\partial y} = \frac{-x^2}{\left(y - 2x^2\right)^2} \Rightarrow \frac{\partial f}{\partial y}(1,1) = \frac{-1^2}{\left(1 - 2(1)^2\right)^2} = -1$$

donc : $\operatorname{grad} f(1,1) = (2, -1)$

EXO 4 :

$$f(x,y) = x^2(x+y)$$

calculons : $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$

$$\frac{\partial f}{\partial x} = 2x(x+y) + x^2, \quad \frac{\partial f}{\partial y} = x^2.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial x}\left(2x(x+y) + x^2\right).$$

$$= 2(x+y) + 2x + 2x$$

$$= 2x + 2y + 4x = 6x + 2y.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial y}\left(x^2\right) = 0.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial f}{\partial x}\left(x^2\right) = 2x.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial f}{\partial y}\left(2x(x+y)+x^2\right)$$

$$= 2x.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

2) $g(x,y) = e^{xy}$.

$$\frac{\partial g}{\partial x} = y e^{xy}, \quad \frac{\partial g}{\partial y} = x e^{xy}.$$

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial g}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial g}{\partial x}\left(y e^{xy}\right) = y^2 e^{xy}.$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial g}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial g}{\partial y}\left(x e^{xy}\right) = x^2 e^{xy}.$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial g}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial g}{\partial x}\left(x e^{xy}\right)$$

$$= e^{xy} + yx\, e^{xy} = e^{xy}(1+xy).$$

$$\frac{\partial^2 g}{\partial y \partial x} = \frac{\partial g}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial g}{\partial y}\left(y e^{xy}\right)$$

$$= e^{xy} + xy\, e^{xy} = e^{xy}(1+xy)$$

EXO 5 :

$$I_1 = \int_1^2 \int_{-1}^1 \frac{x^2}{y}\, dx\, dy.$$

$$= \int_1^2 \frac{1}{y}\left[\int_{-1}^1 x^2\, dx\right] dy = \int_1^2 \frac{1}{y}\left[\frac{x^3}{3}\right]_{-1}^1 dy$$

$$= \left(\frac{1}{3} - \frac{(-1)^3}{3}\right)\int_1^2 \frac{1}{y}\, dy = \frac{2}{3}\left[\ln|y|\right]_1^2$$

$$= \frac{2}{3}\left(\ln|2| - \cancel{\ln|1|}^{\,0}\right) = \frac{2\ln(2)}{3}.$$

$$I_2 = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x+y)\, dx\, dy.$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \left[\sin(x)\cos(y) + \cos(x)\sin(y)\right] dx\, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin(x)\cos(y)\, dx\, dy + \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos(x)\sin(y)\, dx\, dy.$$

$$= \int_0^{\frac{\pi}{2}} \sin(x)\left(\int_0^{\frac{\pi}{2}} \cos(y)\, dy\right) dx + \int_0^{\frac{\pi}{2}} \cos(x)\left[\int_0^{\frac{\pi}{2}} \sin(y)\, dy\right] dx.$$

$$= \int_0^{\frac{\pi}{2}} \sin(x)\left(\sin y\Big|_0^{\frac{\pi}{2}}\right) dx + \int_0^{\frac{\pi}{2}} \cos(x)\left(-\cos(y)\Big|_0^{\frac{\pi}{2}}\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \sin(x)\,(1-0)\, dx + \int_0^{\frac{\pi}{2}} \cos(x)\,(0-(-1))\, dx$$

$$= -\cos x\Big|_0^{\frac{\pi}{2}} + \sin x\Big|_0^{\frac{\pi}{2}} = 0-(-1) + (1-0) = 2.$$

$$I_3 = \iint_D (x^2+y^2)\, dx\, dy .$$

$$D = \{ (x,y) \in \mathbb{R}^2 \mid x,y \geq 0 , \quad x+y \leq 1 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \quad 0 \leq y \leq 1-x \}.$$

$$I_3 = \int_0^1 \left(\int_0^{1-x} x^2+y^2 \, dy \right) dx .$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 x^2(1-x) + \frac{(1-x)^3}{3} \, dx .$$

$$= \frac{1}{3} \int_0^1 (1-x)\left(3x^2 + (1-x)^2\right) dx$$

$$= \frac{1}{3} \int_0^1 -4x^3 + 6x^2 - 3x + 1 \, dx$$

$$= \frac{1}{3} \left[-4 \frac{x^4}{4} + 6 \frac{x^3}{3} - 3 \frac{x^2}{2} + x \right]_0^1 = \frac{1}{6} .$$

$$I_4 = \iint_D \cos(xy)\, dx\, dy .$$

$$D = \{ (x,y) \in \mathbb{R}^2 / \ 1 \leq x \leq 2, \quad 0 \leq xy \leq 1 \}$$

$$= \{ (x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 , \quad 0 \leq y \leq \frac{1}{x} \}$$

$$I_4 = \int_1^2 \left(\int_0^{\frac{1}{x}} \cos(xy)\, dy \right) dx = \int_1^2 \left[\frac{1}{x} \sin(xy) \right]_0^{\frac{2}{x}} dx$$

$$= \int_1^2 \frac{1}{x}\left(\sin\left(2.\frac{2}{2}\right) - \cancel{\sin(0)}^{0}\right) dx$$

$$= \sin(2) \int_1^2 \frac{1}{x}\, dx = \sin(2)\left[\ln|x|\right]_1^2$$

$$= \sin(2)\left(\ln(2) - \cancel{\ln(1)}^{0}\right) = \ln(2)\sin(2).$$

$$I_5 = \iint_D (x+y)\, dx\, dy.$$

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq y,\ x^2+y^2 \leq 1\}.$$

$$= \left\{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq \frac{1}{\sqrt{2}},\ x \leq y \leq \sqrt{1-x^2}\right\}$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \left[\int_x^{\sqrt{1-x^2}} (x+y)\, dy\right] dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \left(xy + \frac{y^2}{2}\right)_x^{\sqrt{1-x^2}} dx.$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x\sqrt{1-x^2} + \frac{1-x^2}{2} - x^2 - \frac{x^2}{2}\, dx.$$

$$= \underbrace{\int_0^{\frac{1}{\sqrt{2}}} x\sqrt{1-x^2}\, dx}_{J} + \underbrace{\int_0^{\frac{1}{\sqrt{2}}} -2x^2 + \frac{1}{2}\, dx}_{K}$$

$$K = \int_0^{\frac{1}{\sqrt{2}}} -2x^2 + \frac{1}{2}\, dx = -2\frac{x^3}{3} + \frac{1}{2}x \Big|_0^{\frac{1}{\sqrt{2}}}$$

$$= -\frac{2}{3}\left(\frac{1}{\sqrt{2}}\right)^3 + \frac{1}{2}\,\frac{1}{\sqrt{2}} = \frac{1}{6\sqrt{2}}$$

$$J = \int_0^{\frac{1}{\sqrt{2}}} x\sqrt{1-x^2}\, dx$$

soit $u = x^2 \Rightarrow x = \sqrt{u}$

$$dx = \frac{1}{2\sqrt{u}}\, du$$

$x = 0 \Rightarrow u = 0$

$x = \frac{1}{\sqrt{2}} \Rightarrow u = \frac{1}{2}$

$$J = \int_0^{\frac{1}{2}} \cancel{\sqrt{u}} \cdot \sqrt{1-u} \cdot \frac{1}{2\cancel{\sqrt{u}}}\, du.$$

$$= \frac{1}{2}\int_0^{\frac{1}{2}} (1-u)^{\frac{1}{2}}\, du = \frac{1}{2}\left[\frac{1}{\frac{1}{2}+1}(1+u)^{\frac{1}{2}+1}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\frac{1}{\frac{3}{2}}(1+u)^{\frac{3}{2}}\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{3}\left(1+\frac{1}{2}\right)^{\frac{1}{2}} - 0.$$

$$= \frac{1}{3}\left(\frac{3}{2}\right)^{\frac{1}{2}} = \frac{1}{3}\left(\frac{1}{2}\right)^{\frac{1}{2}} \cdot \frac{1}{2} = \frac{1}{6\sqrt{2}}$$

$$I = K + J = \frac{1}{6\sqrt{2}} + \frac{1}{6\sqrt{2}} = \frac{2}{6\sqrt{2}} = \frac{1}{3\sqrt{2}}$$

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