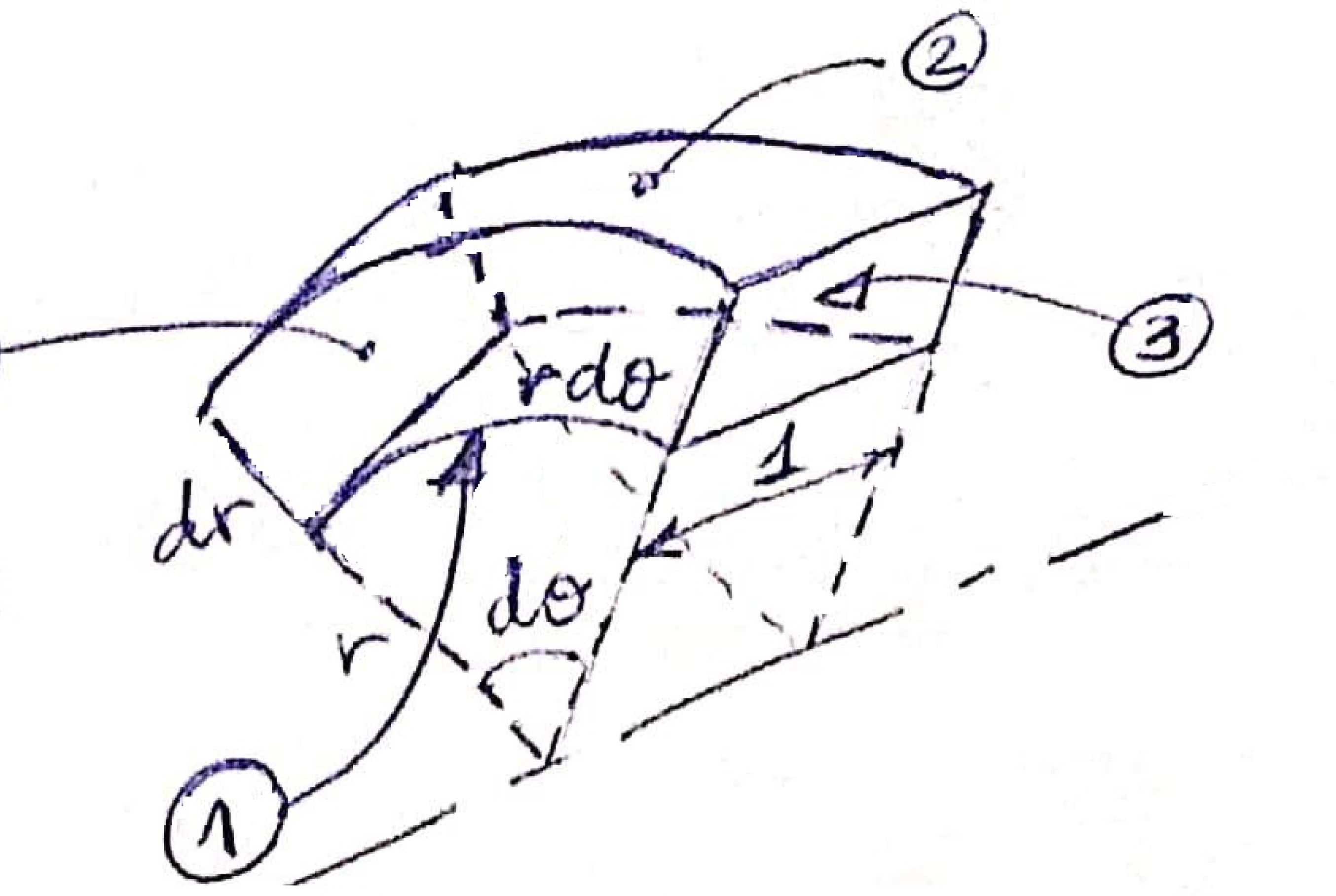
EXO 1 :



* Le taux d'accumulation de ϕ :

Quantité de la variable ϕ à l'instant t :

$$dm\,\phi$$

Quantité de la variable ϕ à l'instant $t+\delta t$: $dm\,\phi + \frac{\partial}{\partial t}(dm\phi)\delta t$

alors le taux d'accumulation :

$$\frac{[dm\phi + \frac{\partial}{\partial t}(dm\phi)\delta t - dm\phi]}{\delta t} = \frac{\partial}{\partial t}(dm\,\phi)$$

$$\frac{\partial}{\partial t}(dm\phi) = \frac{\partial}{\partial t}(\phi\,\rho\, r\,dr\,d\theta \times 1m) = \frac{\partial}{\partial t}(\rho\, r\,dr\,d\theta\,\phi)$$

* Les flux convectifs :

- Le flux convectif entrant par la face ① : $\rho.\phi.U_r.r.d\theta$
- " " sortant " " " ② : $\rho\phi U_r\, rd\theta + \frac{\partial}{\partial r}(\rho\phi U_r\, rd\theta)dr$
- " " entrant " " " ③ : $\rho\phi U_\theta\, dr$
- " " sortant " " " ④ : $\rho\phi U_\theta dr + \frac{\partial}{r\partial\theta}(\rho\phi U_\theta dr)\, rd\theta$

* Les flux diffusifs :

- le flux diffusif entrant par la face ① : $-\Gamma_\phi \frac{\partial\phi}{\partial r} rd\theta$
- " " " " " ② : $-\Gamma_\phi \frac{\partial\phi}{\partial r} rd\theta + \frac{\partial}{\partial r}(-\Gamma_\phi \frac{\partial\phi}{\partial r} rd\theta)dr$
- " " " " " ③ : $-\Gamma_\phi \frac{\partial\phi}{r\partial\theta} dr$
- " " " " " ④ : $-\Gamma_\phi \frac{\partial\phi}{r\partial\theta} dr + \frac{\partial}{r\partial\theta}(-\Gamma_\phi \frac{\partial\phi}{r\partial\theta} dr)\, rd\theta$

$$\frac{\partial}{\partial t}(\rho\, r dr d\theta\,\phi) = -\frac{\partial}{\partial r}(\rho\phi U_r\, rd\theta)dr - \frac{\partial}{r\partial\theta}(\rho\phi U_\theta dr)\, rd\theta + \frac{\partial}{\partial r}(\Gamma_\phi \frac{\partial\phi}{\partial r} rd\theta)dr + \frac{\partial}{r\partial\theta}(\Gamma_\phi \frac{\partial\phi}{r\partial\theta}dr)\, rd\theta$$

$$\frac{\partial}{\partial t}(\rho\phi) + \frac{1}{r\partial r}(\rho\phi U_r r) + \frac{\partial}{r\partial\theta}(\rho\phi U_\theta) = \frac{\partial}{r\partial r}\left(r\Gamma_\phi \frac{\partial\phi}{\partial r}\right) + \frac{\partial}{r\partial\theta}\left(\Gamma_\phi \frac{\partial\phi}{r\partial\theta}\right)$$

Exo 2 :

Le taux d'accumulation de ϕ :

La quantité de la variable ϕ à l'instant t est : $dm\,\phi$

La qté de la variable ϕ à l'instant $t + \delta t$ est : $dm\,\phi + \frac{\partial}{\partial t}(dm\,\phi)\,\delta t$

Alors le taux d'accumulation est :

$$\frac{\left[dm\,\phi + \frac{\partial}{\partial t}(dm\,\phi)\,\delta t - dm\,\phi\right]}{\delta t} = \frac{\partial}{\partial t}(dm\,\phi)$$

$$\frac{\partial}{\partial t}(dm\,\phi) = \frac{\partial}{\partial t}(\phi\,\rho\,r^2 \sin\theta\, dr\, d\varphi\, d\theta)$$

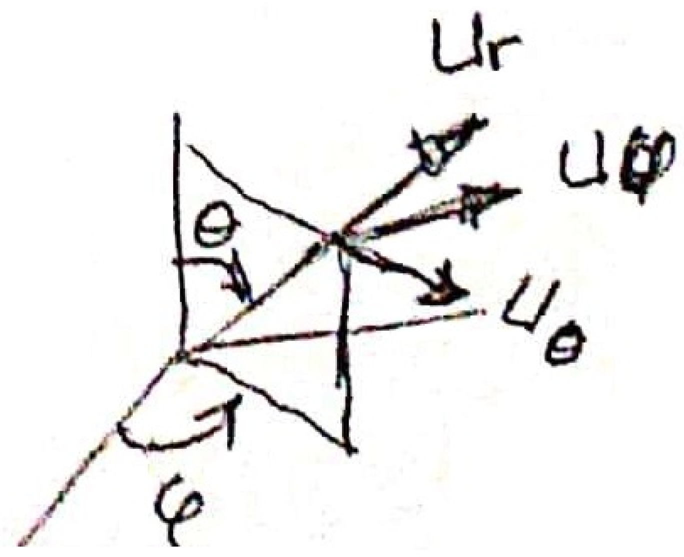
* Les flux convectifs :

- entrant par la face ① :

$\rho\,\phi\,U_r\,r^2 \sin\theta\, d\theta\, d\varphi$

- Sortant par la face ②

$$(\qquad) + \frac{\partial}{\partial r}(\rho\,\phi\,U_r\,r^2 \sin\theta\, d\theta\, d\varphi)\,dr$$

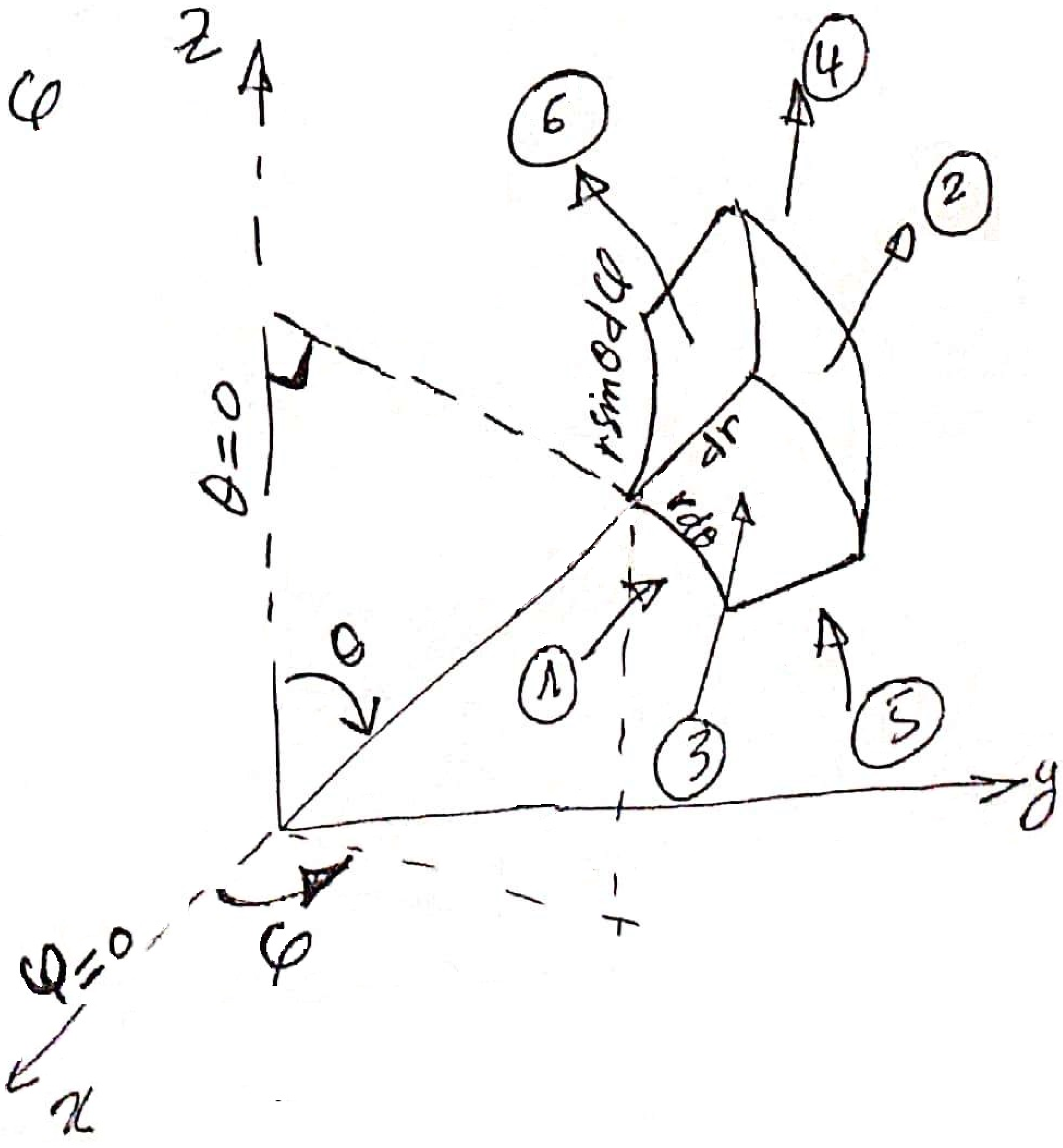


* Entrant par la face ③ :

$\rho\,\phi\,U_\varphi\,r\,dr\,d\theta$

* Sortant par la face ④ :

$$(\qquad) + \frac{\partial}{r\sin\theta\, d\varphi}(\qquad)\,r\sin\theta\, d\varphi$$



* Entrant par la face ⑤ :

$\rho\,\phi\,U_\theta\,r\sin\theta\, d\varphi\, dr$

* Sortant par la face ⑥ :

$$(\qquad) + \frac{\partial}{r\,\partial\theta}(\qquad)\,r\,d\theta$$

Le taux d'accumulation nette par :

① : $-\frac{\partial}{r^2\partial r}(\rho\,\phi\,U_r\,r^2)\,r^2\sin\theta\, d\theta\, d\varphi\, dr$

② : $-\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}(\rho\,\phi\,U_\varphi)\,r^2\sin\theta\, dr\, d\theta\, d\varphi$

③ : $-\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\,\phi\,U_\theta \sin\theta)\,r^2\sin\theta\, dr\, d\theta\, d\varphi$

Flux diffusifs :

entrant par la face ① : $-\Gamma_\phi \dfrac{\partial \phi}{\partial r} r^2 \sin\theta \, d\theta \, d\varphi$

Sortant " " ② : $(\quad) + \dfrac{\partial}{\partial r}(\quad) dr$

Entrant par la face ③ : $-\Gamma_\phi \dfrac{\partial \phi}{r \sin\theta \, d\varphi} r \, dr \, d\theta$

Sortant par la face ④ : $(\quad) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \varphi}(\quad) r \sin\theta \, d\varphi$

Entrant par la face ⑤ : $-\Gamma_\phi \dfrac{\partial \phi}{r \partial \theta} r \sin\theta \, dr \, d\varphi$

Sortant par la face ⑥ : $(\quad) + \dfrac{\partial}{r \partial \theta}(\quad) r \, d\theta$

taux de diffusion entrant nette par :

① : $\dfrac{\partial}{\partial r}\left(\Gamma_\phi \dfrac{\partial \phi}{\partial r} r^2 \sin\theta \, d\theta \, d\varphi\right) dr = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(\Gamma_\phi \dfrac{\partial \phi}{\partial r} r^2\right) r^2 \sin\theta \, d\theta \, d\varphi \, dr$

② : $\dfrac{\partial}{\partial \varphi}\left(\Gamma_\phi \dfrac{\partial \phi}{r \sin\theta \, \partial \varphi} r \, dr \, d\theta\right) d\varphi = \dfrac{1}{r^2 \sin^2\theta}\left(\Gamma_\phi \dfrac{\partial \phi}{\partial \varphi}\right) r^2 \sin\theta \, d\theta \, d\varphi \, dr$

③ : $\dfrac{\partial}{\partial \theta}\left(\Gamma_\phi \dfrac{\partial \phi}{\partial \theta} \sin\theta \, dr \, d\varphi\right) d\theta = \dfrac{1}{r^2 \sin\theta}\left(\Gamma_\phi \dfrac{\partial \phi}{\partial \theta} \sin\theta\right) r^2 \sin\theta \, d\theta \, d\varphi \, dr$

Donc l'équation de transport s'écrit comme suit :

$$\frac{\partial}{\partial t}(\rho \phi) + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho \phi U_r \cdot r^2) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \varphi}(\rho \phi U_\varphi) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho \phi U_\theta \sin\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\Gamma_\phi \frac{\partial \phi}{\partial r} r^2\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \varphi}\left(\Gamma_\phi \frac{\partial \phi}{\partial \varphi}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\Gamma_\phi \frac{\partial \phi}{\partial \theta} \sin\theta\right)$$

Exo 3 :

Pour un écoulement incompressible 2D, les éqts de qté de mvt suivant x et y s'écrivent comme suit :

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad ----(1)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad ---(2)$$

En dérrivant l'éqt ① par rapport à y et en dérrivant l'éqt ② par rapport à x, on obtient :

$$\rho\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \quad ----(3)$$

$$\rho\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)+\mu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \quad ----(4)$$

La soustraction de ③ et ④ donne :

$$\rho\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\right]=\cancel{-\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)}$$

$$+\mu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\cancel{\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)}-\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)-\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)-\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)\right]=$$

$$\mu\left[\frac{\partial^3 v}{\partial x^3}+\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial y^2}\right)-\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}\right)-\frac{\partial^3 u}{\partial y^3}\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-u\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial x\partial y}$$

$$-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}-v\frac{\partial^2 u}{\partial y^2}=\frac{\mu}{\rho}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial x}\cancel{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}^{0}-\frac{\partial u}{\partial y}\cancel{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}^{0}$$

$$=\nu\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]$$

Sachant que : * l'éqt de continuité est : $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

* La vorticité est : $\vec{\xi}=\vec{\nabla}\wedge\vec{V}$

donc la composante suivant z est : $\xi_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

Donc : $\frac{\partial \xi_z}{\partial t}+u\frac{\partial \xi_z}{\partial x}+v\frac{\partial \xi_z}{\partial y}=\nu\left(\frac{\partial^2 \xi_z}{\partial x^2}+\frac{\partial^2 \xi_z}{\partial y^2}\right)\rightarrow\frac{d\xi_z}{dt}=\mu\nabla^2\xi_z$

Pour un fluide parfait ($\nu=0$) $\rightarrow \frac{d\xi_z}{dt}=0$

Pour un écoulement (2D) d'un fluide parfait, le changement de vorticité d'une particule de fluide lorsqu'elle se déplace dans le champ d'écoulement est nul.

Exo 4 :

Les hypothèses simplificatrices sont :

* Écoulement stationnaire $\left(\frac{\partial}{\partial t}=0\right)$
* " Incompressible $(\rho = c^{te})$
* Il n'ya ni écoulement, ni variation des propriétés suivant z $\left(V_z=0 \text{ et } \frac{\partial}{\partial z}=0\right)$
* L'écoulement est symétrique donc il n'ya pas une variation suivant θ $\left(\frac{\partial}{\partial \theta}=0\right)$
* Pas de frottement $(\mu=0)$
* $g_r = g_\theta = g_z = 0$
* Vortex libre (Ecoulement dans le coude)

Les éqts de q^{te} de mouvement deviennent :

suivant z : $\frac{\partial P}{\partial z}=0$

Suivant θ : $\frac{\partial P}{\partial \theta}=0$

→ La pression est en fonction de r seulement

suivant r : $\frac{\partial P}{\partial r}=\frac{dP}{dr}=\frac{\rho V_\theta^2}{r}$ avec $V_\theta = V = \frac{c}{r}$

$$dP = \rho\frac{V^2}{r}dr \longrightarrow \int_{P_1}^{P_2} dP = \int_{R_1}^{R_2}\frac{\rho c^2}{r^3}dr$$

$$\Delta P = P_2 - P_1 = \frac{\rho c^2}{2}\left(\frac{1}{R_1^2}-\frac{1}{R_2^2}\right) \rightarrow \Delta P = \frac{\rho c^2 (R_2^2 - R_1^2)}{2R_1^2 R_2^2} \quad \dots (1)$$

Le débit est défini comme suit : $dQ = V dA$

$$\text{Donc: } Q = \int_A V dA = \int_{R_1}^{R_2} V L\, dr = L\int_{R_1}^{R_2}\frac{c}{r}dr = L.c \ln\left(\frac{R_2}{R_1}\right)$$

$$\text{or : } c = \frac{Q}{L\ln\left(\frac{R_2}{R_1}\right)} \quad \dots (2)$$

En utilisant l'eqt (1) : $\Delta P = P_2 - P_1 = \frac{\rho c^2 (R_2^2 - R_1^2)}{2 R_1^2 R_2^2}$

$$\Delta P = \frac{\rho Q^2 (R_2^2 - R_1^2)}{2L^2\left[\ln\left(\frac{R_2}{R_1}\right)\right]^2 . R_1^2 . R_2^2} \longrightarrow Q^2 = \frac{\Delta P . 2L^2\left[\ln\left(\frac{R_2}{R_1}\right)\right]^2 . R_1^2 . R_2^2}{\rho (R_2^2 - R_1^2)}$$

$$Q = L\ln\left(\frac{R_2}{R_1}\right)\sqrt{\frac{2R_1^2 R_2^2}{\rho(R_2^2-R_1^2)}}\sqrt{\Delta P} = K\sqrt{\Delta P}$$

$$\text{où } K = L\ln\left(\frac{R_2}{R_1}\right)\sqrt{\frac{2R_2^2 R_1^2}{\rho(R_2^2 - R_1^2)}}$$